國立政治大學 應用數學系

碩士 學位論文

(s,t) 單色矩形於二色棋盤之存在性

The Existence of (s,t)-Monochromatic-rectangles in a 2-colored Checkerboard

碩士班學生：卓駿焰　撰
指導教授：張宜武　博士

中華民國 105 年 6 月 30 日

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業經本委員會審議通過
論文考試委員會委員：

指導教授：

系主任：

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中文摘要

本文藉由矩形棋盤著色探討完全二分圖 $K_{m,n}$ 由兩種顏色任意塗邊，使得此兩色著邊之完全二分圖 $K_{m,n}$ 會包含單色子圖 $K_{s,2}$、$K_{s,3}$ 與 $K_{s,t}$ （$s \geq 2$），我們將討論參數 n 與 s 之間滿必須滿足何種關係。本文也將介紹處理棋盤著色問題的一般方法與技巧，以及透過棋盤如何將棋盤問題轉化為圖論問題，並且將它推廣。

關鍵字：完全二分圖、單色子圖

Abstract

In this paper, we study the two edge-coloring of $K_{m,n}$ such that $K_{m,n}$ contains a monochromatic subgraph $K_{s,2}$, $K_{s,3}$ or $K_{s,t}$. We find the relation between n, s by investigating a two coloring of a checkerboard.

Keywords: Complete bipartite graph;Monochromatic subgraph

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Chapter 1

Introduction

We often encounter problems related the Ramsey numbers [1] in many Mathematical Competitions of High School Students. In this thesis, we use the ideas of the Ramsey numbers on the checkerboard problems. We follow [2] for the notations in graph theory and the definitions of the complete bipartite graph $K_{m,n}$, and follow [3] to construct the correspondence between the checkerboards and complete bipartite graphs. Jiong-Sheng Li provides the minimal sizes of k-colored square checkerboards which have monochromatic-rectangles in [3], we define the generalized monochromatic-rectangles and discuss the existence of such rectangles in an $m \times n$ checkerboard. In this thesis, we only consider the checkerboards which are arbitrarily colored by two colors and we called it two-colored checkerboard.

In the second chapter, we discuss the minimal columns of the 2-colored checkerboard which has (2,2)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems. In the third chapter, we extend the second chapter to discuss the minimal columns of the 2-colored checkerboard which has (2,t)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems.

In the forth chapter, we discuss the minimal columns of the 2-colored checkerboard which has (3,2)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems. In the fifth chapter, we extend the forth chapter to discuss the minimal columns of the 2-colored checkerboard which has (3,t)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems.

In the second chapter to the fifth chapter, all results have been proved in [4], but we improve the proof such that be more general and we also propose some amendments in the third chapter. In the last two chapter, we propose the generalized conclusions. We discuss the minimal columns of the 2-colored checkerboard which has (s,t)-monochromatic-rectangles by fixing the rows. At the end, we convert the results to graphic problems.

By [3] we convert the grids of a checkerboard into the edges of a complete bipartite graph, the number of rows and columns correspond to the number of vertices in complete partite sets, X and Y, respectively.

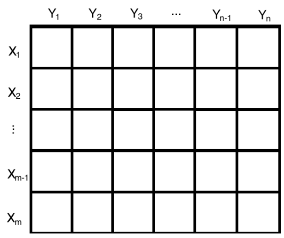


Figure 1.1: $m \times n$ checkerboard

If the grid in the i-th row and the j-th column of the checkerboard is black, then the correspond edge $x_i y_j$ in the correspond complete bipartite graph is solid. And the white grid is correspond the dashed edge. The following is a 2-colored $m \times n$ checkerboard correspond to a 2-coloring $K_{m,n}$.

Figure 1.2: A 3×4 checkerboard and the correspond complete bipartite graph.

Chapter 2

(2,2)-Monochromatic-rectangles in a Checkerboard

Definition 2.1. An $m \times n$ rectangle is called a **(s,t)-monochromatic-rectangle**, if in first column there are s grids including the first one and the last one that have the same color, and there are other t-1 columns including the last column that are copies of the first column.

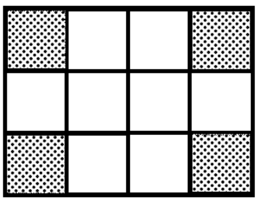


Figure 2.1: A (2,2)-monochromatic-rectangle

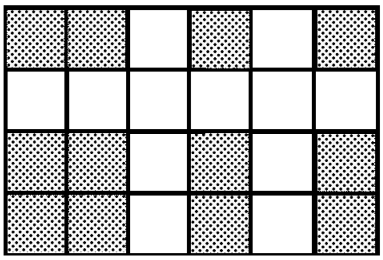


Figure 2.2: A (3,4)-monochromatic-rectangle

2.1 The Case of $2 \times n$ Checkerboard

If there are two (2,1)-monochromatic-rectangles of the same color, then the checkerboard has a (2,2)-monochromatic rectangle. Otherwise, there is no (2,2)-monochromatic rectangle.

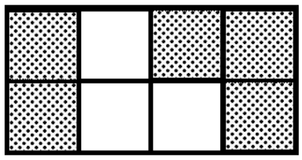


Figure 2.3: There are two (2,1)-monochromatic-rectangles of the same color in the checkerboard.

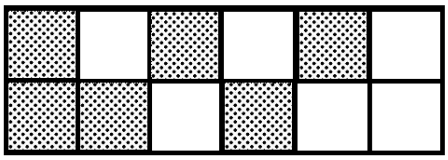
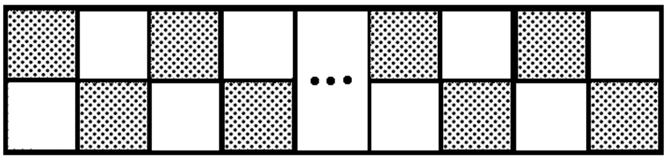


Figure 2.4: There is no two (2,1)-monochromatic-rectangles of the same color in the checkerboard.

Therefore, in a 2-colored $2 \times n$ checkerboard a (2,2)-monochromatic-rectangle may not exist.



2.2 The Case of $3 \times n$ Checkerboard

Definition 2.2. In a checkerboard, two (s,1)-monochromatic-rectangles of the same color are **the same**, if one is a copy of the other one.

Definition 2.3. An $n \times 1$ column **contains** a (s,1)-monochromatic-rectangle means that there are s grids of the same color in the column.

Note: An $n \times 1$ column contains at most C_s^n distinct (s,1)-monochromatic-rectangles.

We consider

Lemma 2.4. *In every 2-colored $3 \times n$ checkerboard, $n = 7$ is the smallest number such that there exists a (2,2)-monochromatic-rectangle.*

Proof. To prove that we need to exhibit a 2-colored 3×6 checkerboard that has no (2,2)-monochromatic-rectangles. In a column, there are at most C_2^3 distinct black (2,1)-monochromatic-rectangles. So we can distribute the C_2^3 distinct black (2,1)-monochromatic-rectangles and the C_2^3 distinct white (2,1)-monochromatic-rectangles to the 6 columns, then the 2-colored 3×6 checkerboards have no (2,2)-monochromatic-rectangles.

By pigeonhole principle, there are at least $\lceil \frac{3 \times 7}{2} \rceil = 11$ grids of the same color. Without loss of generality, let the color be black. Then we have $\sum_{i=1}^{7} d_i \geq 11$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored 3×7 checkerboard has a coloring such that there is no (2,2)-monochromatic-rectangle, then any two columns don't contain the same black (2,1)-monochromatic-rectangles, each column contains $C_2^{d_i}$ distinct black (2,1)-monochromatic-rectangles, and the total number of distinct black (2,1)-monochromatic-rectangles is not more than C_2^3. So we have

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_7} \leq C_2^3 \tag{2.2.1}$$

Let $d_1 + d_2 + \cdots + d_7 = 11 + t$, where $t \in \mathbb{N}$. Then we can transform (2.2.1) to

$$d_1^2 + d_2^2 + \cdots + d_7^2 \leq 17 + t \tag{2.2.2}$$

By Cauchy–Schwarz inequality, we get

$$(d_1^2 + d_2^2 + \cdots + d_7^2)(1^2 + 1^2 + \cdots + 1^2) \geq (d_1 + d_2 + \cdots + d_7)^2$$
$$\Rightarrow (17 + t) \times 7 \geq (11 + t)^2$$

So we have

$$t^2 + 15t + 2 \leq 0$$

But $t \in \mathbb{N}$, we reach a contradiction in the last inequality. Therefore, every 2-coloring of 3×7 checkerboard yields a (2,2)-monochromatic-rectangle. □

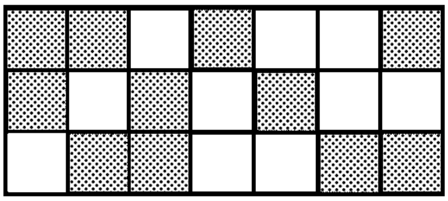


Figure 2.5: Every 2-colored 3×7 checkerboard contains a (2,2)-monochromatic-rectangle.

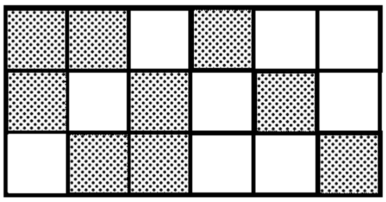


Figure 2.6: There is a 2-colored 3×6 checkerboard containing no (2,2)-monochromatic-rectangles.

2.3 The Case of $4 \times n$ Checkerboard

Lemma 2.5. *In every 2-colored $4 \times n$ checkerboard, $n = 7$ is the smallest number such that there exists a (2,2)-monochromatic-rectangle.*

Proof. By Lemma2.4, in every 2-colored 3×7 checkerboard, there is a (2,2)-monochromatic-rectangle. Therefore, in every 2-colored 4×7 checkerboard, there is a (2,2)-monochromatic-rectangle. □

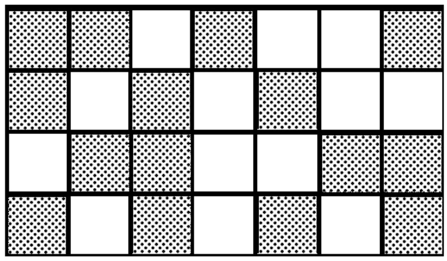


Figure 2.7: Every 2-colored 4×7 checkerboard contains a (2,2)-monochromatic-rectangle.

2.4 The Case of $5 \times n$ Checkerboard

Lemma 2.6. *In every 2-colored $5 \times n$ checkerboard, $n = 5$ is the smallest number such that there exists a (2,2)-monochromatic-rectangle.*

*Proof.*To prove that we need to exhibit a 2-colored 5×4 checkerboard that has no (2,2)-monochromatic-rectangles. By the Lemma2.5, we have a 2-colored 4×5 checkerboard which doesn't have (2,2)-monochromatic-rectangles. We rotate the checkerboard, so we have the 2-colored 5×4 checkerboard that has no (2,2)-monochromatic-rectangles.

By pigeonhole principle, there are at least $\lceil \frac{5 \times 5}{2} \rceil = 13$ grids of the same color. Without loss of generality, let the color be black. Then we have $\sum_{i=1}^{5} d_i \geq 13$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume two colored 5×5 checkerboard has a coloring such that there is no (2,2)-monochromatic-rectangles, then any two columns don't contain the same black (2,1)-monochromatic-rectangles, each column contains $C_2^{d_i}$ distinct black (2,1)-monochromatic-rectangles, and the total number of distinct black (2,1)-monochromatic-rectangles is not more than C_2^5. So we have

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_5} \leq C_2^5 \tag{2.4.1}$$

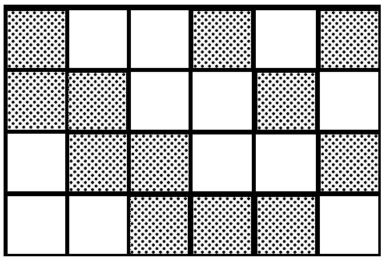


Figure 2.8: There is a 2-colored 4×6 checkerboard containing no (2,2)-monochromatic-rectangles.

Let $d_1 + d_2 + \cdots + d_5 = 13 + t$, where $t \in \mathbb{N}$. Then we can transform (2.4.1) to

$$d_1^2 + d_2^2 + \cdots + d_5^2 \leq 23 + t \tag{2.4.2}$$

By Cauchy–Schwarz inequality, we get

$$\begin{aligned}(d_1^2 + d_2^2 + \cdots + d_5^2)(1^2 + 1^2 + \cdots + 1^2) &\geq (d_1 + d_2 + \cdots + d_5)^2 \\ \Rightarrow (23 + t) \times 5 &\geq (13 + t)^2\end{aligned}$$

So we have

$$t^2 + 21t + 54 \leq 0$$

But $t \in \mathbb{N}$, we reach a contradiction in the last inequality. Therefore, every 2-colored 5×5 checkerboard yields a (2,2)-monochromatic-rectangle. □

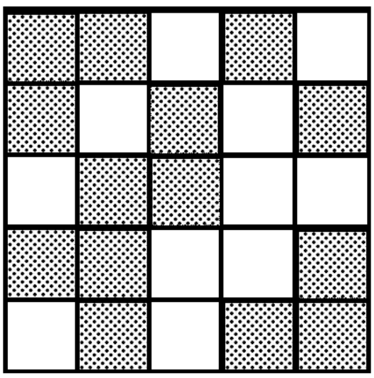


Figure 2.9: Every 2-colored 5×5 checkerboard contains a (2,2)-monochromatic-rectangle.

2.5 Summary

The case of $m \times n$ checkerboard, where $m \geq 6$, can be obtained by rotation of the rectangles. For example, 6×5 checkerboard can be considered to 5×6 checkerboard, so it has a (2,2)-monochromatic-rectangle.

We can convert the above theorems to graphic problems. We have the following proposition.

- If $n > 6$, every 2-coloring of $K_{3,n}$ contains a monochromatic $K_{2,2}$.
- If $n > 6$, every 2-coloring of $K_{4,n}$ contains a monochromatic $K_{2,2}$.
- If $n > 4$, every 2-coloring of $K_{5,n}$ contains a monochromatic $K_{2,2}$.

Example 2.7. Every 2-coloring of $K_{3,7}$ exists a monochromatic $K_{2,2}$ subgraph.

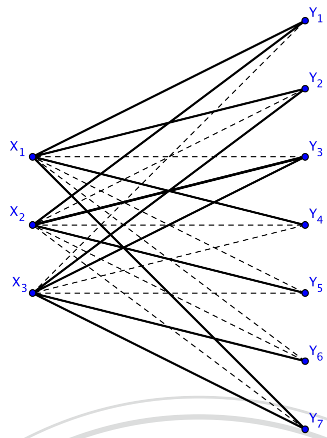
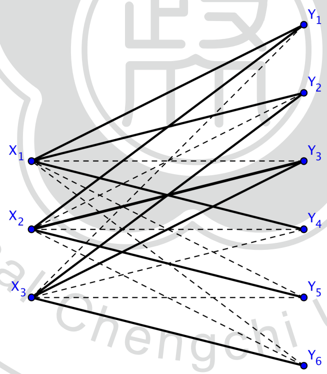


Figure 2.10: The subgraph induced by $\{X_1, X_3, Y_2, Y_7\}$ is a monochromatic copy of $K_{2,2}$.

Example 2.8. There is a 2-coloring of $K_{3,6}$ no a monochromatic $K_{2,2}$ subgraph.



Example 2.9. Every 2-coloring of $K_{4,7}$ exists a monochromatic $K_{2,2}$ subgraph.

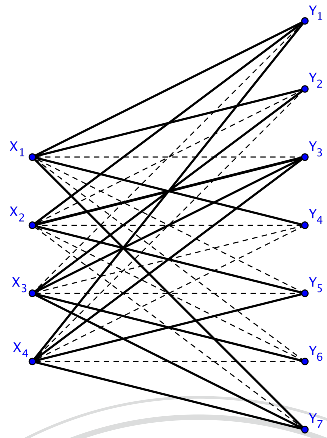
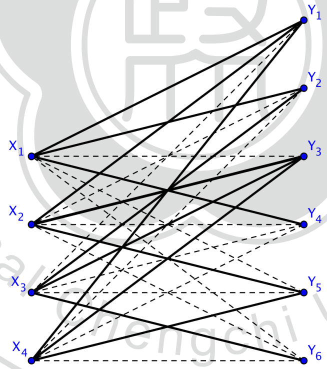


Figure 2.11: The subgraph induced by $\{X_1, X_3, Y_2, Y_7\}$ is a monochromatic copy of $K_{2,2}$.

Example 2.10. There is a 2-coloring of $K_{4,6}$ no a monochromatic $K_{2,2}$ subgraph.



Example 2.11. Every 2-coloring of $K_{5,5}$ exists a monochromatic $K_{2,2}$ subgraph.

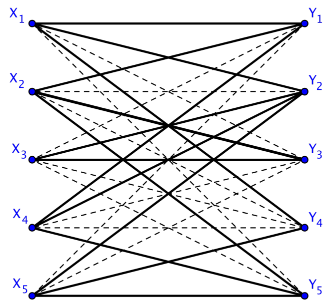


Figure 2.12: The subgraph induced by $\{X_4, X_5, Y_2, Y_5\}$ is a monochromatic copy of $K_{2,2}$.

Chapter 3

(2,t)-monochromatic-rectangles in a Checkerboard

3.1 The Case of $2 \times n$ Checkerboard

If there are t (2,1)-monochromatic-rectangles of the same color, then the checkerboard has a (2,t)-monochromatic-rectangle. Otherwise, there is no (2,t)-monochromatic-rectangle.

3.2 The Case of $3 \times n$ Checkerboard

Theorem 3.1. *If $n > (6t-6)$, where $t \geq 2$, then in every 2-colored $3 \times n$ checkerboard. There is a (2,t)-monochromatic-rectangle.*

Proof. If $n = (6t-6)+1 = 6t-5$ (Only prove that every coloring of two colors $3 \times (6t-5)$ checkerboard, there is a (2,t)-monochromatic-rectangle.) By pigeonhole principle, there are at least $\lceil \frac{6t-5}{2} \rceil = 3t-2$ columns that have at least two of same color grids. Without loss of generality, let the color be black. Then $d_i \geq 2 \quad i = 1, 2, \ldots, (3t-2)$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume two colored $3 \times (6t-5)$ checkerboard has a coloring such that there is no (2,t)-monochromatic-rectangles, then any t columns don't contain the same black (2,1)-monochromatic-rectangles, each column contains $C_2^{d_i}$ distinct black (2,1)-monochromatic-rectangles, and the total number of distinct black (2,1)-

monochromatic-rectangles is not more than $(t-1) \cdot C_2^3$. So we have,

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{3t-2}} \leq (t-1) \cdot C_2^3$$

Because $d_i \geq 2$

$$C_2^2 + C_2^2 + \cdots + C_2^2 \leq C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{3t-2}}$$

combining the two results shows

$$C_2^2 + C_2^2 + \cdots + C_2^2 \leq (t-1) \cdot C_2^3$$
$$\Rightarrow 3t - 2 \leq 3t - 3$$

$1 \leq 0$ we reach a contradiction. So, If $n > (6t-6)$, where $t \geq 2$, then in every 2-coloring of $3 \times n$ checkerboard. There is a (2,t)-monochromatic-rectangle. □

3.3 The Case of $4 \times n$ Checkerboard

Theorem 3.2. *If $n > (6t-6)$, where $s \geq 2$, then in every 2-colored $4 \times n$ checkerboard. There is a (2,t)-monochromatic-rectangle.*

Proof. By Theorem 3.1, in every 2-colored $3 \times (6t-6)$ checkerboard, there is a (2,t)-monochromatic-rectangle. Therefore, in every 2-colored $4 \times (6t-6)$ checkerboard, there is a (2,t)-monochromatic-rectangle. □

3.4 The Case of $5 \times n$ Checkerboard

Theorem 3.3. *If $n > (5t-6)$, where $t \geq 2$, then in every 2-colored $5 \times n$ checkerboard. There is a (2,t)-monochromatic-rectangle, where t is even. And $C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_n} > (2t-3) \cdot C_2^5$ where d_i is the number of black grids of the i^{th} column of the checkerboard.*

Proof. Suppose $t = 2k - 2$, where k is integer greater than two, we use induction on k, If $n = (10k-16) + 1 = 10k - 15$ (Only prove that every coloring of two colors $5 \times (10k-15)$

checkerboard, there is a $(2k-2,2)$-monochromatic-rectangle.

Basis step: When $k=2$, by the Lemma 2.6 we have in every 2-coloring of 5×5 checkerboard, there is a (2,2)-monochromatic-rectangle. Suppose $k=s$ is true, $s\geq 2$ and $s\in\mathbb{N}$ for all 2-coloring of $5\times(10s-15)$ checkerboard, there is a $(2,2s-2)$-monochromatic-rectangle. By pigeonhole principle, there are at least $\lceil\frac{5\times(10s-15)}{2}\rceil = 25s-37$ grids of the same color. Without loss of generality, let the color be black. So, we have $\sum_{i=1}^{10s-15} d_i = 25s-37$, and $C_2^{d_1}+C_2^{d_2}+\cdots+C_2^{d_{10s-15}} > (2s-3)\cdot C_2^5$

Induction step: When $k=s+1$, $n=10(s+1)-15=10s-5$, by pigeonhole principle, there are at least $\lceil\frac{5\times(10s-5)}{2}\rceil = 25s-12$ grids of the same color in $5\times(10s-5)$ checkerboard. Without loss of generality, let the color be black. We have $\sum_{i=1}^{10s-5} d_i = 25s-12$. By induction hypothesis, $\sum_{i=11}^{10s-5} d_i = 25s-37$, so $\sum_{i=1}^{10} d_i = 25$ and $C_2^{d_i}$ is the number of black (2,1)-monochromatic-rectangles in the i^{th} column, $i=1,2,\cdots,(10s-5)$. Assume two colored $5\times(10s-5)$ checkerboard has a coloring such that there is no $(2,2s)$-monochromatic-rectangles, then any $2s$ columns don't contain the same black (2,1)-monochromatic-rectangles, each column contains $C_2^{d_i}$ distinct black (2,1)-monochromatic-rectangles, and the total number of distinct black (2,1)-monochromatic-rectangles is not more than $(2s-1)\cdot C_2^5$. So we have

$$C_2^{d_1}+C_2^{d_2}+\cdots+C_2^{d_{10s-5}} \leq (2s-1)\cdot C_2^5 \tag{3.4.1}$$

By induction hypothesis

$$C_2^{d_{11}}+C_2^{d_{12}}+\cdots+C_2^{d_{10s-5}} > (2s-3)\cdot C_2^5$$

So, we have

$$C_2^{d_1}+C_2^{d_2}+\cdots+C_2^{d_{10}} < (2s-1)\cdot C_2^5-(2s-3)\cdot C_2^5$$

$$C_2^{d_1}+C_2^{d_2}+\cdots+C_2^{d_{10}} < 20$$

By the definition of combination we get

$$\frac{d_1(d_1-1)}{2}+\frac{d_2(d_2-1)}{2}+\cdots+\frac{d_{10}(d_{10}-1)}{2}<20$$

multiple 2 in both side

$$(d_1^2+d_2^2+\cdots+d_{10}^2)-(d_1+d_2+\cdots+d_{10})<40$$

$$d_1^2+d_2^2+\cdots+d_{10}^2<65$$

By Cauchy–Schwarz inequality

$$(d_1^2+d_2^2+\cdots+d_{10}^2)\cdot 10\geq(d_1+d_2+\cdots+d_{10})^2$$

$$(d_1^2+d_2^2+\cdots+d_{10}^2)\geq\frac{25^2}{10}$$

$$\Longrightarrow 62.5\leq(d_1^2+d_2^2+\cdots+d_{10}^2)\leq 64$$

$\because d_i$ are integer $\therefore (d_1^2+d_2^2+\cdots+d_{10}^2)$ must be 63 or 64
$Case1: d_1^2+d_2^2+\cdots+d_{10}^2=63$

$$d_2^2+d_3^2+\cdots+d_{10}^2=63-d_1^2$$

and

$$d_2+d_3+\cdots+d_{10}=25-d_1$$

By Cauchy–Schwarz inequality

$$(d_2^2 + d_3^2 + \cdots + d_{10}^2) \cdot 9 \geq (d_2 + d_3 + \cdots + d_{10})^2$$

$$(63 - d_1^2) \cdot 9 \geq (25 - d_1)^2$$

$$10d_1^2 - 50d_1 + 58 \leq 0$$

$$1.83 \leq d_1 \leq 3.17$$

and $d_1$is integer, so d_1 must be 2 or 3. Similarly, $d_2, d_3, \cdots, d_{10}$ must be 2 or 3 But $d_1 + d_2 + \cdots + d_{10} = 25$, so $d_1, d_2, \cdots, d_{10}$ consists of five $2's$ and five $3's$ Therefore, $d_1^2 + d_2^2 + \cdots + d_{10}^2 = 65 \neq 63$, we reach a contradiction.
$Case2 : d_1^2 + d_2^2 + \cdots + d_{10}^2 = 64$

$$d_2^2 + d_3^2 + \cdots + d_{10}^2 = 64 - d_1^2$$

and

$$d_2 + d_3 + \cdots + d_{10} = 25 - d_1$$

By Cauchy–Schwarz inequality

$$(d_2^2 + d_3^2 + \cdots + d_{10}^2) \cdot 9 \geq (d_2 + d_3 + \cdots + d_{10})^2$$

$$(64 - d_1^2) \cdot 9 \geq (25 - d_1)^2$$

$$10d_1^2 - 50d_1 + 49 \leq 0$$

$$1.34 \leq d_1 \leq 3.66$$

and $d_1$is integer, so d_1 must be 2 or 3. Similarly, $d_2, d_3, \cdots, d_{10}$ must be 2 or 3. But $d_1 + d_2 + \cdots + d_{10} = 25$, so $d_1, d_2, \cdots, d_{10}$ consists of five $2's$ and five $3's$. Therefore, $d_1^2 + d_2^2 + \cdots + d_{10}^2 = 65 \neq 64$, we reach a contradiction.

So for all 2-colored $5 \times (10s - 5)$ checkerboard, there is a (2,2s)-monochromatic rectangle.

By induction, $\forall k \geq 2$ and $k \in \mathbb{N}$, in every 2-colored $5 \times (5t - 6)$ checkerboard, there is a $(2, t)$-monochromatic rectangle, where $t = 2k - 2$.

Lemma 3.4. *In every 2-colored* 5×11 *checkerboard, there is a (2,3)-monochromatic-rectangle.*

Proof. By pigeonhole principle, there are at least $\lceil \frac{5 \times 11}{2} \rceil = 28$ grids of the same color. Without loss of generality, let the color be black. Then we have $\sum_{i=1}^{11} d_i \geq 28$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume two colored 5×11 checkerboard has a coloring such that there is no monochromatic-rectangles, then any three columns don't contain the same black (2,1)-monochromatic-rectangles, each column contains $C_2^{d_i}$ distinct black (2,1)-monochromatic-rectangles, and the total number of distinct black (2,1)-monochromatic-rectangles is not more than $2 \cdot C_2^5$. So we have

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{11}} \leq 2 \times C_2^5 \tag{3.4.2}$$

Let $d_1 + d_2 + \cdots + d_{11} = 28 + t$, where $t \in \mathbb{N}$. Then we can transform (3.4.2) to

$$d_1^2 + d_2^2 + \cdots + d_{11}^2 \leq 68 + t \tag{3.4.3}$$

By Cauchy–Schwarz inequality, we get

$$(d_1^2 + d_2^2 + \cdots + d_{11}^2)(1^2 + 1^2 + \cdots + 1^2) \geq (d_1 + d_2 + \cdots + d_{11})^2$$
$$\Rightarrow (68 + t) \times 11 \geq (28 + t)^2$$

So we have

$$t^2 + 45t + 36 \leq 0$$

But $t \in \mathbb{N}$, the last inequality is contradiction. Therefore, every 2-colored 5×11 checkerboard yields a (2,3)-monochromatic-rectangle. $\square$

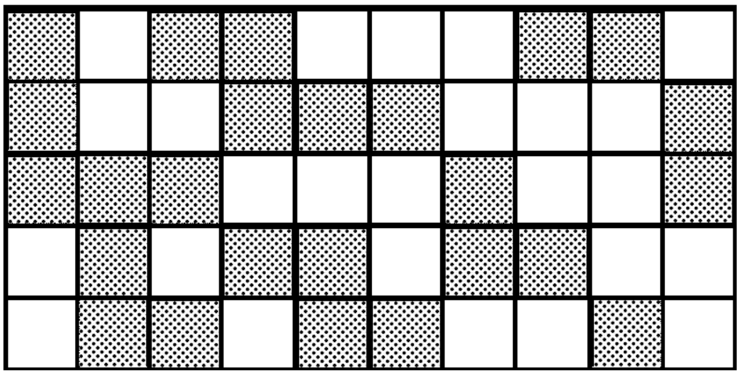


Figure 3.1: There is a 2-colored 5×10 checkerboard containing no a (2,3)-monochromatic-rectangle.

Theorem 3.5. *If $n > (5t-5)$, where $t \geq 2$, then in every 2-colored $5 \times n$ checkerboard. There is a (2,t)-monochromatic-rectangle, where s is odd. And $C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_n} > (2t-2) \cdot C_2^5$ where d_i is the number of black grids of the i^{th} column of the checkerboard.*

Proof. Suppose $t = 2k-1$, where k is integer greater than two, we use induction on k, If $n = (10k-10)+1 = 10k-9$ (Only prove that every 2-colored $5 \times (10k-9)$ checkerboard, there is a $(2, 2k-1)$-monochromatic-rectangle.

Basis step: When $k = 2$, by the Lemma 3.4 we have in every 2-colored 5×11 checkerboard, there is a (2,3)-monochromatic-rectangle. Suppose $k = s$ is true, $s \geq 2$ and $s \in \mathbb{N}$ for all 2-colored $5 \times (10s-9)$ checkerboard, there is a $(2, 2s-1)$-monochromatic-rectangle. By pigeonhole principle, there are at least $\lceil \frac{5 \times (10s-9)}{2} \rceil = 25s-22$ grids of the same color. Without loss of generality, let the color be black. So, we have $\sum_{i=1}^{10t-9} d_i = 25s-22$, and $C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{10s-9}} > (2s-2) \cdot C_2^5$

Induction step: When $k = s+1$, $n = 10(s+1)-9 = 10s+1$, by pigeonhole principle, there are at least $\lceil \frac{5 \times (10s+1)}{2} \rceil = 25s+3$ grids of the same color in $5 \times (10s+1)$ checkerboard. Without loss of generality, let the color be black. We have $\sum_{i=1}^{10s+1} d_i = 25s+3$. By induction hypothesis, $\sum_{i=11}^{10s-5} d_i = 25s-22$, so $\sum_{i=1}^{10} d_i = 25$ and $C_2^{d_i}$ is the number of black-bars in the i^{th} column, $i = 1, 2, \cdots, (10s+1)$. Assume two colored $5 \times (10s+1)$ checkerboard has a coloring such that there is no $(2, 2s+1)$-monochromatic-rectangles, then any $2s+1$ columns don't contain the same black (2,1)-monochromatic-rectangles, each column contains $C_2^{d_i}$ distinct black (2,1)-monochromatic-rectangles, and the total number of distinct black (2,1)-monochromatic-rectangles is not more than $2s \cdot C_2^5$. So we have

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{10s+1}} \leq 2s \cdot C_2^5 \tag{3.4.4}$$

By induction hypothesis

$$C_2^{d_{11}} + C_2^{d_{12}} + \cdots + C_2^{d_{10s+1}} > (2s-2) \cdot C_2^5$$

So, we have

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{10}} < 2s \cdot C_2^5 - (2s-2) \cdot C_2^5$$

$$C_2^{d_1} + C_2^{d_2} + \cdots + C_2^{d_{10}} < 20$$

By the definition of combination we get

$$\frac{d_1(d_1-1)}{2} + \frac{d_2(d_2-1)}{2} + \cdots + \frac{d_{10}(d_{10}-1)}{2} < 20$$

multiple 2 in both side

$$(d_1^2 + d_2^2 + \cdots + d_{10}^2) - (d_1 + d_2 + \cdots + d_{10}) < 40$$

$$d_1^2 + d_2^2 + \cdots + d_{10}^2 < 65$$

By Cauchy–Schwarz inequality

$$(d_1^2 + d_2^2 + \cdots + d_{10}^2) \cdot 10 \geq (d_1 + d_2 + \cdots + d_{10})^2$$

$$(d_1^2 + d_2^2 + \cdots + d_{10}^2) \geq \frac{25^2}{10}$$

$$\implies 62.5 \leq (d_1^2 + d_2^2 + \cdots + d_{10}^2) \leq 64$$

$\because d_i$ are integer $\therefore (d_1^2 + d_2^2 + \cdots + d_{10}^2)$ must be 63 or 64

$Case1 : d_1^2 + d_2^2 + \cdots + d_{10}^2 = 63$

$$d_2^2 + d_3^2 + \cdots + d_{10}^2 = 63 - d_1^2$$

and

$$d_2 + d_3 + \cdots + d_{10} = 25 - d_1$$

By Cauchy–Schwarz inequality

$$(d_2^2 + d_3^2 + \cdots + d_{10}^2) \cdot 9 \geq (d_2 + d_3 + \cdots + d_{10})^2$$

$$(63 - d_1^2) \cdot 9 \geq (25 - d_1)^2$$

$$10d_1^2 - 50d_1 + 58 \leq 0$$

$$1.83 \leq d_1 \leq 3.17$$

and $d_1$is integer, so d_1 must be 2 or 3. Similarly, $d_2, d_3, \cdots, d_{10}$ must be 2 or 3 But $d_1 + d_2 + \cdots + d_{10} = 25$, so $d_1, d_2, \cdots, d_{10}$ consists of five $2's$ and five $3's$ Therefore, $d_1^2 + d_2^2 + \cdots + d_{10}^2 = 65 \neq 63$, we reach a contradiction. $Case2 : d_1^2 + d_2^2 + \cdots + d_{10}^2 = 64$

$$d_2^2 + d_3^2 + \cdots + d_{10}^2 = 64 - d_1^2$$

and

$$d_2 + d_3 + \cdots + d_{10} = 25 - d_1$$

By Cauchy–Schwarz inequality

$$(d_2^2 + d_3^2 + \cdots + d_{10}^2) \cdot 9 \geq (d_2 + d_3 + \cdots + d_{10})^2$$

$$(64 - d_1^2) \cdot 9 \geq (25 - d_1)^2$$

$$10d_1^2 - 50d_1 + 49 \leq 0$$

$$1.34 \leq d_1 \leq 3.66$$

and $d_1$is integer, so d_1 must be 2 or 3. Similarly, $d_2, d_3, \cdots, d_{10}$ must be 2 or 3. But $d_1 + d_2 + \cdots + d_{10} = 25$, so $d_1, d_2, \cdots, d_{10}$ consists of five $2's$ and five $3's$. Therefore, $d_1^2 + d_2^2 + \cdots + d_{10}^2 = 65 \neq 64$, we reach a contradiction.

So for all 2-colored $5 \times (10s - 5)$ checkerboard, there is a $(2, 2s + 1)$-monochromatic rectangle.

By induction, $\forall k \geq 2$ and $k \in \mathbb{N}$, in every 2-colored $5 \times (5t - 5)$ checkerboard, there is a $(2, t)$-monochromatic rectangle,where $t = 2k - 1$. $\square$

3.5 Summary

We can convert the above theorems to graphic problems. We have the following proposition. Let $s \geq 2$

- If $n > 6(t - 1)$, every 2-coloring of $K_{3,n}$ exists a monochromatic $K_{2,t}$ subgraph.
- If $n > 6(t - 1)$, every 2-coloring of $K_{4,n}$ exists a monochromatic $K_{2,t}$ subgraph.
- If $n > (10t - 16)$, every 2-coloring of $K_{5,n}$ exists a monochromatic $K_{2,(2t-2)}$ subgraph.
- If $n > (10t - 10)$, every 2-coloring of $K_{5,n}$ exists a monochromatic $K_{2,(2t-1)}$ subgraph.

Chapter 4

(3,2)-Monochromatic-rectangles in a Checkerboard

4.1 The Case of $3 \times n$ Checkerboard

If there are two columns of all grids are of the same color, then the checkerboard has a (3,2)-monochromatic-rectangle. Otherwise, there is no (3,2)-monochromatic-rectangle.

4.2 The Case of $4 \times n$ Checkerboard

If every column of a $4 \times n$ checkerboard has two black grids and two white grids, then it doesn't have s a (3,2)monochromatic-rectangle.Therefore, for every two color $4 \times n$ checkerboard exists a coloring such that has no (3,2)-monochromatic-rectangles in the $4 \times n$ checkerboard.

4.3 The Case of $5 \times n$ Checkerboard

Lemma 4.1. *In every 2-colored* $5 \times n$ *checkerboard,* $n = 21$ *is the smallest number such that there exists a (3,2)-monochromatic-rectangle.*

*Proof.*To prove that we need to exhibit a two colored 5×20 checkerboard has no (3,2)-

monochromatic-rectangles. In a column, there are at most C_3^5 distinct black (3,1)-monochromatic-rectangles. So we can distribute the C_3^5 distinct black (3,1)-monochromatic-rectangles and the C_3^5 distinct white (3,1)-monochromatic-rectangles to the 20 columns, then the two colored 5×20 checkerboards have no (3,2)-monochromatic-rectangles.

By pigeonhole principle, there are at least $\lceil \frac{21}{2} \rceil = 11$ columns with at least three grids of the same color. Without loss of generality, let the color be black. Then $d_i \geq 3 \quad i = 1, 2, \ldots, 11$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored 5×21 checkerboard has a coloring such that there is no (3,2)monochromatic-rectangles, then any two columns don't contain the same black (3,1)-monochromatic-rectangles, each column contains $C_3^{d_i}$ distinct black (3,1)-monochromatic-rectangles, and the total number of distinct black (3,1)-monochromatic-rectangles is not more than C_3^5. So we have

$$C_3^{d_1} + C_3^{d_2} + \cdots + C_3^{d_{11}} \leq C_3^5$$

Because $d_i \geq 3$

$$C_3^3 + C_3^3 + \cdots + C_3^3 \leq C_3^{d_1} + C_3^{d_2} + \cdots + C_3^{d_{11}}$$

combining the two results shows

$$\begin{aligned} C_3^3 + C_3^3 + \cdots + C_3^3 &\leq C_3^5 \\ \Rightarrow 11 &\leq 10 \end{aligned}$$

$11 \leq 10$ iwe reach a contradiction in the last inequality. So, If $n > 21$, then in every 2-coloring of $5 \times n$ checkerboard. There is a (3,2)monochromatic-rectangle. □

4.4 The Case of $6 \times n$ Checkerboard

Lemma 4.2. *In every 2-colored $6 \times n$ checkerboard, $n = 21$ is the smallest number such that there exists a (3,2)-monochromatic-rectangle.*

Proof. By Lemma4.1, in every 2-colored 5×21 checkerboard, there is a (3,2)monochromatic-

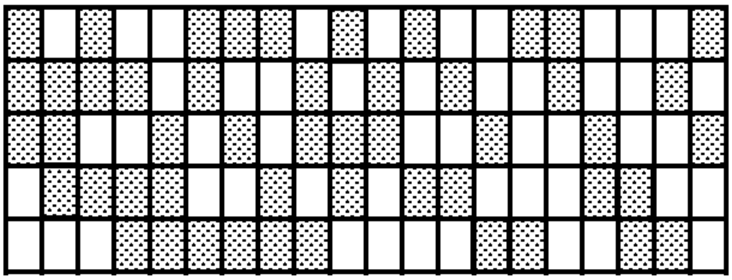


Figure 4.1: There is a 2-colored 5×20 checkerboard containing no a (3,2)-monochromatic-rectangle.

rectangle. Therefore, in every 2-colored 6×21 checkerboard, there is a (3,2)monochromatic-rectangle. □

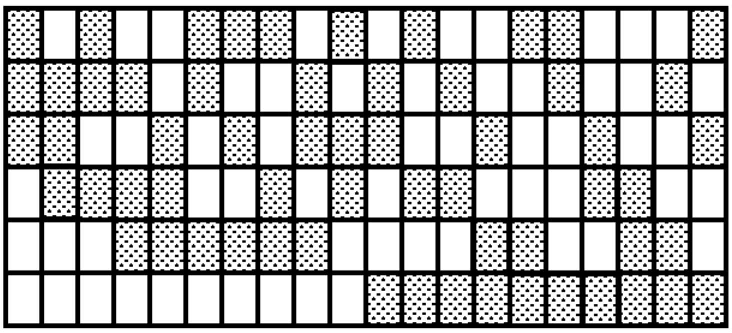


Figure 4.2: There is a 2-colored 6×20 checkerboard containing no a (3,2)-monochromatic-rectangle.

4.5 Summary

We can convert the above theorems to graphic problems. We have the following proposition.

- If $n > 20$, every 2-coloring of $K_{5,n}$ contains a monochromatic $K_{3,2}$.
- If $n > 20$, every 2-coloring of $K_{6,n}$ contains a monochromatic $K_{3,2}$.

Chapter 5

(3,t)-Monochromatic-rectangles in a Checkerboard

5.1 The Case of $5 \times n$ Checkerboard

Theorem 5.1. *If $n > 20(t-1)$, where $t \geq 2$, then in every 2-colored $5 \times n$ checkerboard, there is an (3,t)-Monochromatic-rectangle.*

Proof. If $n = (20t - 20) + 1 = 20t - 19$ (Only prove that every 2-colored $5 \times (20t - 19)$ checkerboard, there is a s-monochromatic-rectangle.) By pigeonhole principle, there are at least $\lceil \frac{20t-19}{2} \rceil = 10t - 9$ columns that have at least three of same color grids. Without loss of generality, let the color be black. Then $d_i \geq 3 \quad i = 1, 2, \ldots, (10t - 9)$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored $3 \times (20t - 19)$ checkerboard has a coloring such that there is no (3,t)-Monochromatic-rectangles, then any s columns don't contain the same black (3,1)-monochromatic-rectangles, each column contains $C_3^{d_i}$ distinct black (3,1)-monochromatic-rectangles, and the total number of distinct black (3,1)-monochromatic-rectangles is not more than $(t-1) \cdot C_2^3$. So we have,

$$C_3^{d_1} + C_3^{d_2} + \cdots + C_3^{d_{10t-9}} \leq (t-1) \cdot C_3^5$$

Because $d_i \geq 3$

$$C_3^3 + C_3^3 + \cdots + C_3^3 \leq C_3^{d_1} + C_3^{d_2} + \cdots + C_3^{d_{10t-9}}$$

combining the two results shows

$$C_3^3 + C_3^3 + \cdots + C_3^3 \leq (t-1) \cdot C_3^5$$
$$\Rightarrow 10t - 9 \leq 10t - 10$$

$1 \leq 0$ we reach a contradiction in the last inequality. So, If $n > (20t - 20)$, where $t \geq 2$, then in every 2-colored $5 \times n$ checkerboard. There is an (3,t)-Monochromatic-rectangle. □

5.2 The Case of $6 \times n$ Checkerboard

Theorem 5.2. *If $n > 20(t-1)$, where $s \geq 2$, then in every 2-colored $6 \times n$ checkerboard, there is a (3,t)-Monochromatic-rectangle.*

Proof. By Theorem 5.1, in every 2-colored $5 \times (20t - 20)$ checkerboard, there is a (3,t)-Monochromatic-rectangle . Therefore, in every 2-colored $6 \times (20t - 20)$ checkerboard, there is a (3,t)-Monochromatic-rectangle. □

5.3 Summary

We can convert the above theorems to graphic problems. We have the following proposition. Let $s \geq 2$

- If $n > 20(t-1)$, every 2-coloring of $K_{5,n}$ exists a monochromatic $K_{3,t}$ subgraph.
- If $n > 20(t-1)$, every 2-coloring of $K_{6,n}$ exists a monochromatic $K_{3,t}$ subgraph.

Chapter 6

(s,2)-Monochromatic-rectangles in a Checkerboard

6.1 The Case of $(2s-2)\times n$ Checkerboard

If every column of a $(2s-2)\times n$ checkerboard has s-1 black grids and s-1 white grids, then it doesn't have s a (s,2)-monochromatic-rectangle.Therefore, for every two color $2s-2\times n$ checkerboard exists a coloring such that has no (s,2)-monochromatic-rectangle in the $(2s-2)\times n$ checkerboard.

6.2 The Case of $(2s-1)\times n$ Checkerboard

Lemma 6.1. *In every 2-colored $(2s-1)\times n$ checkerboard, $n=(2C_s^{2s-1}+1)$ is the smallest number such that there exists a (2,2)-monochromatic-rectangle.*

*Proof.*To prove that we need to exhibit a 2-colored $(2s-1)\times 2C_s^{2s-1}$ checkerboard that has no (2,2)-monochromatic-rectangles. In a column, there are at most C_s^{2s-1} distinct black (s,1)-monochromatic-rectangles. So we can distribute the C_s^{2s-1} distinct black (s,1)-monochromatic-rectangles and the C_s^{2s-1} distinct white (s,1)-monochromatic-rectangles to the $2C_s^{2s-1}$ columns, then the 2-colored $(2s-1)\times 2C_s^{2s-1}$ checkerboards have no (s,2)-monochromatic-rectangles. By pigeonhole principle, there are at least $\lceil\frac{2C_s^{2s-1}+1}{2}\rceil = C_s^{2s-1}+1$ columns with at least s grids are of the same color. Without loss of generality, let the color be black. Then $d_i \geq s \quad i = 1, 2, \ldots, C_s^{2s-1} + 1$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored $2s - 1 \times 2C_s^{2s-1} + 1$ checkerboard has a coloring such that there is no (s,2)monochromatic-rectangles, then any two columns don't contain the same black (s,1)-monochromatic-rectangles, each column contains $C_s^{d_i}$ distinct black (s,1)-monochromatic-rectangles, and the total number of distinct black (s,1)-monochromatic-rectangles is not more than C_s^{2s-1}. So we have

$$\sum_{k=1}^{C_s^{2s-1}+1} C_s^{d_k} \leq C_s^{2s-1}$$

Because $d_i \geq s$

$$C_s^s + C_s^s + \cdots + C_s^s \leq \sum_{k=1}^{C_s^{2s-1}+1} C_s^{d_k}$$

combining the two results shows

$$C_s^s + C_s^s + \cdots + C_s^s \leq C_s^{2s-1}$$
$$\Rightarrow C_s^{2s-1} + 1 \leq C_s^{2s-1}$$

$1 \leq 0$ we reach a contradiction in the last inequality. So, If $n > 2C_s^{2s-1} + 1$, then in every 2-colored $(2s-1) \times (2C_s^{2s-1} + 1)$ checkerboard. There is a (s,2)-monochromatic-rectangle. □

6.3 The Case of $2s \times n$ Checkerboard

Lemma 6.2. *In every 2-colored $(2s) \times (2C_s^{2s-1} + 1)$ checkerboard, there is a (s,2)monochromatic-rectangle.*

Proof. By Lemma6.1, in every 2-colored $(2s-1) \times (2C_s^{2s-1} + 1)$ checkerboard, there is a (s,2)monochromatic-rectangle. Therefore, in every 2-colored $(2s) \times (2C_s^{2s-1} + 1)$ checkerboard, there is a (s,2)monochromatic-rectangle. □

6.4 Summary

We can convert the above theorems to graphic problems. We have the following proposition.

- If $n > (2C_s^{2s-1} + 1)$, every 2-coloring of $K_{(2s-1),n}$ contains a monochromatic $K_{s,2}$.
- If $n > (2C_s^{2s-1} + 1)$, every 2-coloring of $K_{2s,n}$ contains a monochromatic $K_{s,2}$.

Chapter 7

(s,t)-Monochromatic-rectangles in a Checkerboard

7.1 The Case of $(2s-1)\times n$ Checkerboard

Theorem 7.1. *In every 2-colored $(2s-1)\times(2(t-1)C_s^{2s-1}+1)$ checkerboard, there is a (s,t)monochromatic-rectangle.*

Proof. By pigeonhole principle, there are at least $\lceil\frac{2(t-1)C_s^{2s-1}+1}{2}\rceil = (t-1)C_s^{2s-1}+1$ columns with at least s grids are of the same color. Without loss of generality, let the color be black. Then $d_i \geq t \;\; i=1,2,\ldots,(t-1)C_s^{2s-1}+1$, where d_i is the number of black grids of the i^{th} column of the checkerboard. Assume 2-colored $2s-1\times 2(t-1)C_s^{2s-1}+1$ checkerboard has a coloring such that there is no (s,t)monochromatic-rectangle, then any s columns don't contain the same black (s,1)-monochromatic-rectangles, each column contains $C_s^{d_i}$ distinct black (s,1)-monochromatic-rectangles, and the total number of distinct black (s,1)-monochromatic-rectangles is not more than $(t-1)C_s^{2s-1}$. So we have

$$\sum_{k=1}^{(t-1)C_s^{2s-1}+1} C_s^{d_k} \leq (t-1)C_s^{2s-1}$$

Because $d_i \geq s$

$$C_s^s + C_s^s + \cdots + C_s^s \leq \sum_{k=1}^{(t-1)C_s^{2s-1}+1} C_s^{d_k}$$

combining the two results shows

$$\begin{aligned} C_s^s + C_s^s + \cdots + C_s^s &\leq (t-1)C_s^{2s-1} \\ \Rightarrow (t-1)C_s^{2s-1} + 1 &\leq (t-1)C_s^{2s-1} \end{aligned}$$

$1 \leq 0$ we reach a contradiction in the last inequality. So, If $n > 2(t-1)C_s^{2s-1} + 1$, then in every 2-colored $(2s-1) \times (2(t-1)C_s^{2s-1} + 1)$ checkerboard. There is a (s,t)-monochromatic-rectangle. □

7.2 The Case of $2s \times n$ Checkerboard

Theorem 7.2. *In every 2-colored $(2s) \times (2(t-1)C_s^{2s-1} + 1)$ checkerboard, there is a (s,t)monochromatic-rectangle.*

Proof. By Theorem7.1, in every 2-colored $(2s-1) \times (2(t-1)C_s^{2s-1} + 1)$ checkerboard, there is a (s,t)monochromatic-rectangle. Therefore, in every 2-colored $(2s) \times (2(t-1)C_s^{2s-1} + 1)$ checkerboard, there is a (s,t)monochromatic-rectangle. □

7.3 Summary

We can convert the above theorems to graphic problems. We have the following proposition.

- If $n > (2(t-1)C_s^{2s-1} + 1)$, every 2-coloring of $K_{(2s-1),n}$ contains a monochromatic $K_{s,t}$.
- If $n > (2(t-1)C_s^{2s-1} + 1)$, every 2-coloring of $K_{2s,n}$ contains a monochromatic $K_{s,t}$.

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