國立政治大學應用數學系

碩士學位論文

指導教授：符聖珍 博士

The maximum principles for the partial dynamic operators on time scales

時間刻度下偏動態算子的極大值定理

碩士班學生：陳家盛 撰

中華民國一百年七月

Contents

謝辭

光陰似箭，轉眼間我已經在政大應數生活了三個年頭。回想起剛進政大時的青澀模樣，到現在是個即將畢業的準碩士生，之間的生活回憶著實讓我永難忘懷，心中除了不捨還是不捨。在這裡除了帶走許多專業領域的數學知識外，更讓我感覺驕傲的是結交到許多同甘共苦、共進退的好夥伴、知己，有了你們，才讓我的碩士生活過的如此多采多姿。

在這幾年的碩士生活中，過得最精實大概就屬撰寫論文的這段時間，當然最要感謝的就是我的美女指導教授符聖珍老師。老師讓我知道了做研究應該具備的研究態度，也讓我見識到教育家有教無類的精神，即使是碰到我這種愛耍小聰明、偷懶的學生，依舊是用大量的時間以及心力來指導我。對於老師付出的愛與關懷，我都能感受的到，內心的感謝自然非三言兩語能形容。也感謝陳天進老師，老師在教學上的熱忱，是讓我最敬佩的。也因爲遇見老師，讓我體會到原來眞的存在那種所謂的亦師亦友，甚或是忘年之交的感覺。老師並沒有我印象中教授的那種古板形象，老師直來直往、豪邁的個性讓我感到非常的投緣，所以很多研究室裡或所上的問題也敢直接跟老師溝通，老師也都會想辦法幫我們解決，對於老師的照顧眞的是由衷感激。也很感謝林景隆老師千里迢迢從台南上來當我的口試委員，謝謝老師。

感謝研究室裡大大小小的戰友們，因爲有你們，讓我每天都能上演著互相揶揄調侃、打打殺殺的快樂戲碼。雖然每天這樣打打鬧鬧，但其實都能感覺的出來大家是非常在乎彼此的，也因爲這樣，讓研究室多了許多溫馨歡樂氣氛。謝謝亮哥，你大概是我在這三年中唯一能信服的人，你的客觀分析能力，不僅在數學上、事物上表現都是高人一等，即使在感情上分析也是如此，讓我很懷疑你沒交女朋友這件事的眞實性。雖然很常互嗆(大部分都是你被我攻擊)，但這都是我對你友愛的表現，哈哈，你就別太care吧，我之後出去工作一定會幫你介紹女友的。感謝治陞，你豁達的個性以及對事物的看法大概是最貼近我的人，唯一的不同就

是你比我淫蕩多了。很懷念以前一起夜衝唸書、玩世紀帝國以及出遊的日子，也感謝你在我低潮時開導我和介紹女生給我認識，雖然沒結果(被你羞辱了許久)，但還是感謝你的用心。感謝小貓，你大概是我第2個媽，有時甚至覺得你比我媽還囉唆，但我知道，愛之深、責之切，你一定是爲我好才唸我，但還是注意一下你的表達方式，很多時候我會反擊不是沒原因的，哈。至於吳宥柔學妹，你是我認識所有人中唯一要求我在謝辭中提到妳的人(不知你是哪來的自信)，這著實造成我極大的困擾，整篇謝辭百分之90的時間都在思考著要感謝你什麼事情(非常難想)。嘖嘖，好吧，感謝你盧小小、玻璃心的個性，眞的是讓我碩士生活的最後一年增添了不少樂趣，讓我忘卻許多煩惱，謝謝你啦(不甘不願)。感謝蔡公宗穎，你逗趣的程度讓我每次出遊都想約你一起。雖然有時我會挖洞給你跳，但你都不會跟我計較，肚量之大都可媲美你的肌肉了，尤其你對朋友的情義相挺程度讓我都想給你一個讚字了。感謝財哥在數學領域上給我的幫助，學長的虛懷若谷更是讓我敬佩，學長太強了啦(財哥：想太多，最好是)。也感謝振偉學長及賴哥在Latex上給我的幫助，讓我在撰寫論文時少花了許多摸索的時間，謝謝你們。

感謝我的高中死黨們，即使是在我趕論文生死交關的時期還是持續的約我出遊，就算被我拒絕還是死纏濫打，我也強迫自己認爲你們是怕我壓力太大才在非常時期還不斷邀約，謝謝你們啦。

最後，我要感謝我的家人們長久以來對我的默默付出，尤其是我的爸爸媽媽。謝謝你們多年來的栽培，我愛你們!!!

陳家盛　謹誌于

國立政治大學應用數學所

中華民國一百年七月

Abstract

In this thesis, we establish the maximum principles for the elliptic dynamic operators and parabolic dynamic operators on multi-dimensional time scales, and apply it to obtain some applications. Indeed, we extend the maximum principles on differential equations and difference equations to the so-called dynamic equations.

中文摘要

在這篇論文裡，我們要討論的是在多維度的時間刻度(time scale)下橢圓型動態算子和拋物型動態算子的極大值定理，並藉此得到一些應用。 事實上，我們是將微分方程及差分方程裡的極大值定理推廣至所謂的動態方程中。

1 Introduction

Maximum principles are an important tool in the study of partial differential and difference equations. For example, they can be used to obtain the existence and uniqueness of solutions and to approximate it. Consequently the theory of maximum principles in difference and differential equations has been investigated extensively, see for example [1] and [2] and the references cited therein.

In recent years, the study of dynamic equations on time scales has received a lot of attentions since it not only can unify the calculation of difference and differential equations but also has various applications. In particular, the maximum principles have been established in [4] for the second order ordinary dynamic operator and [5] for the elliptic dynamic operator. Motivated by the above work, in this thesis, we study the maximum principles for the elliptic dynamic operator

$$\mathcal{L}[u] := \sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + B_i u^{\Delta_i} + C_i u^{\nabla_i})$$

and the parabolic dynamic operator

$$L[u] := \sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}}.$$

Our results improve the results in [5].

This thesis is organized as follows. Section 2 contains some basic definitions and the necessary results about time scales. In Section 3, we present the maximum principles for the elliptic dynamic operators. Finally, in Section 4, we establish the maximum principles for the parabolic dynamic operators, and apply it to obtain some useful applications.

2 Preliminary

For completeness, we state some fundamental definitions and results concerning partial dynamic equations on time scales that we will use in the sequel. It can be regarded as a generalization of the one-dimensional case. More details can be found in [6], [7], [8], and [9].

A time scale is an arbitrary nonempty closed subset of $\mathbb{R}$. Throughout this thesis, we denote $I = \{1, 2, \cdots, n\}$, where $n \in \mathbb{N}$, and we assume that $\mathbb{T}_i$, for each $i \in I$, is a time scale and the set

$$\Lambda = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \{t = (t_1, t_2, \cdots, t_n) \mid t_i \in \mathbb{T}_i \text{ for each } i \in I\},$$

defined by the Cartesian product is an n-dimensional time scale.

Definition 2.1 *For each $i \in I$, the mappings σ_i, $\rho_i : \mathbb{T}_i \to \mathbb{T}_i$ defined by*

$$\sigma_i(u) := \begin{cases} \inf\{v \in \mathbb{T}_i \mid v > u\}, & if \;\; u \neq \max \mathbb{T}_i, \\ \max \mathbb{T}_i, & if \;\; u = \max \mathbb{T}_i, \end{cases}$$

and

$$\rho_i(u) := \begin{cases} \sup\{v \in \mathbb{T}_i \mid v < u\}, & if \;\; u \neq \min \mathbb{T}_i, \\ \min \mathbb{T}_i, & if \;\; u = \min \mathbb{T}_i, \end{cases}$$

are called the ith forward and backward jump operators respectively. In this definition, the corresponding graininess functions μ_i, $\nu_i : \mathbb{T}_i \to [0, \infty)$ are defined by

$$\mu_i(u) := \sigma_i(u) - u, \qquad \nu_i(u) := u - \rho_i(u).$$

For convenience, we define the functions $\hat{\sigma}_i$, $\hat{\rho}_i : \Lambda \to \Lambda$ by

$$\hat{\sigma}_i(t) = (t_1, t_2, \cdots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \cdots, t_n),$$

and

$$\hat{\rho}_i(t) = (t_1, t_2, \cdots, t_{i-1}, \rho_i(t_i), t_{i+1}, \cdots, t_n),$$

for any $t \in \Lambda$ and $i \in I$. In addition, if $u : \Lambda \to \mathbb{R}$ is a function, then the functions $u^{\hat{\sigma}_i}$, $u^{\hat{\rho}_i} : \Lambda \to \mathbb{R}$ are defined by

$$u^{\hat{\sigma}_i}(t) = u(\hat{\sigma}_i(t)) \qquad \textit{and} \qquad u^{\hat{\rho}_i}(t) = u(\hat{\rho}_i(t)),$$

for any $t \in \Lambda$ and $i \in I$.

Definition 2.2 *A point t in Λ is said to be i-right dense if $t_i < \max \mathbb{T}_i$ and $\sigma_i(t_i) = t_i$, and i-left dense if $t_i > \min \mathbb{T}_i$ and $\rho_i(t_i) = t_i$. Also, if $\sigma_i(t_i) > t_i$ then t is called i-right scattered, and if $\rho_i(t_i) < t_i$ then t is called i-left scattered. Moreover, we say that t is i-scattered if it is both i-left scattered and i-right scattered, and i-dense if it is both i-left dense and i-right dense.*

Definition 2.3 *For each $i \in I$, let*

$$(\mathbb{T}_i)^{\mathcal{K}} = \begin{cases} \mathbb{T}_i \backslash \max \mathbb{T}_i, & \textit{if } \mathbb{T}_i \textit{ has a left scattered maximum,} \\ \mathbb{T}_i, & \textit{if } \mathbb{T}_i \textit{ has a left dense maximum.} \end{cases}$$

Then we can define

$$\Lambda^{\mathcal{K}} = (\mathbb{T}_1)^{\mathcal{K}} \times (\mathbb{T}_2)^{\mathcal{K}} \times \cdots \times (\mathbb{T}_n)^{\mathcal{K}}.$$

Assume $u : \Lambda \to \mathbb{R}$ is a function and let $t \in \Lambda^{\mathcal{K}}$. Then we define $u^{\Delta_i}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\, [u(\hat{\sigma}_i(t)) - u(t_1, t_2, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n)] - u^{\Delta_i}(t)[\sigma_i(t_i) - s] \,| \le \varepsilon \mid \sigma_i(t_i) - s \mid,$$

for all $s \in (t_i - \delta, t_i + \delta) \cap \mathbb{T}_i$. In this case, we call $u^{\Delta_i}(t)$ the partial delta derivative of u at t with respect to t_i.

In particular, if we choose $n = 1$ in this definition, then u is a single variable function from $\mathbb{T}_1$ into $\mathbb{R}$, and we denote the delta derivative of u at

$t \in (\mathbb{T}_1)^{\mathcal{K}}$ by $u^{\Delta}(t)$. Moreover, we say that u is delta differentiable at t if $u^{\Delta}(t)$ exists for some $t \in (\mathbb{T}_1)^{\mathcal{K}}$.

Definition 2.4 *For each $i \in I$, let*

$$(\mathbb{T}_i)_{\mathcal{K}} = \begin{cases} \mathbb{T}_i \backslash \min \mathbb{T}_i, & \text{if } \mathbb{T}_i \text{ has a right scattered minimum,} \\ \mathbb{T}_i, & \text{if } \mathbb{T}_i \text{ has a right dense minimum.} \end{cases}$$

Then we can define

$$\Lambda_{\mathcal{K}} = (\mathbb{T}_1)_{\mathcal{K}} \times (\mathbb{T}_2)_{\mathcal{K}} \times \cdots \times (\mathbb{T}_n)_{\mathcal{K}}.$$

Assume $u : \Lambda \to \mathbb{R}$ is a function and let $t \in \Lambda_{\mathcal{K}}$. Then we define $u^{\nabla_i}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\mid [u(\hat{\rho}_i(t)) - u(t_1, t_2, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n)] - u^{\nabla_i}(t)[\rho_i(t_i) - s] \mid \leq \varepsilon \mid \rho_i(t_i) - s \mid,$$

for all $s \in (t_i - \delta, t_i + \delta) \cap \mathbb{T}_i$. In this case, we call $u^{\nabla_i}(t)$ the partial nabla derivative of u at t with respect to t_i.

In particular, if we choose $n = 1$ in this definition, then u is a single variable function from $\mathbb{T}_1$ into $\mathbb{R}$, and we denote the nabla derivative of u at $t \in (\mathbb{T}_1)_{\mathcal{K}}$ by $u^{\nabla}(t)$. Moreover, we say that u is nabla differentiable at t if $u^{\nabla}(t)$ exists for some $t \in (\mathbb{T}_1)_{\mathcal{K}}$.

For convenience, we denote the intersection of $\Lambda^{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ by $\Lambda^{\mathcal{K}}_{\mathcal{K}}$, i.e.,

$$\Lambda^{\mathcal{K}}_{\mathcal{K}} = (\mathbb{T}_1)^{\mathcal{K}}_{\mathcal{K}} \times (\mathbb{T}_2)^{\mathcal{K}}_{\mathcal{K}} \times \cdots \times (\mathbb{T}_n)^{\mathcal{K}}_{\mathcal{K}}.$$

Definition 2.5 *Let $\mathbb{T}$ be an arbitrary time scale. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

Definition 2.6 *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of*

$f : \mathbb{T} \to \mathbb{R}$ *provided*

$$F^{\Delta}(t) = f(t) \qquad holds\ for\ all\ t \in \mathbb{T}^{\mathcal{K}}.$$

We then define the integral of f *by*

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \qquad for\ all\ s, t \in \mathbb{T}.$$

Lemma 2.7 *Every rd-continuous function has a delta antiderivative.*

Definition 2.8 *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called ld-continuous provided it is continuous at left-dense points in* $\mathbb{T}$ *and its right-sided limits exist (finite) at right-dense points in* $\mathbb{T}$.

Definition 2.9 *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a nabla antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided*

$$F^{\nabla}(t) = f(t) \qquad holds\ for\ all\ t \in \mathbb{T}_{\mathcal{K}}.$$

We then define the integral of f *by*

$$\int_s^t f(\tau)\nabla\tau = F(t) - F(s) \qquad for\ all\ s, t \in \mathbb{T}.$$

Lemma 2.10 *Every ld-continuous function has a nabla antiderivative.*

Definition 2.11 *Let* $\mathbb{T}$ *be an arbitrary time scale, and* $p : \mathbb{T} \to \mathbb{R}$ *be a function and satisfy*

$$1 - \nu(t)p(t) \neq 0 \qquad for\ all\ t \in \mathbb{T}_{\mathcal{K}}.$$

Then we define the nabla exponential function by

$$\hat{e}_p(t, s) = exp(\int_s^t g(\tau)\nabla\tau) \qquad for\ s, t \in \mathbb{T},$$

where

$$g(\tau) = \begin{cases} p(\tau), & if\ \nu(\tau) = 0, \\ -\dfrac{1}{\nu(\tau)} Log(1 - \nu(\tau)p(\tau)), & if\ \nu(\tau) \neq 0. \end{cases}$$

Lemma 2.12 *Suppose that α is a negative constant and $s, t, u \in \mathbb{T}$, then*

(a) $\hat{e}_\alpha(t,s) > 0$ *and* $\hat{e}_\alpha(t,t) \equiv 1$;

(b) $\hat{e}_\alpha(t,u)\hat{e}_\alpha(u,s) = \hat{e}_\alpha(t,s)$;

(c) $\hat{e}_\alpha^\nabla(t,s) = \alpha \hat{e}_\alpha(t,s)$.

Lemma 2.13 *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a single variable function and let $t \in \mathbb{T}_\mathcal{K}^\mathcal{K}$, then we have the following:*

(a) If f is delta or nabla differentiable at t, then f is continuous at t.

(b) If f is continuous at a right-scattered point t, then f is delta differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(c) If t is right-dense, then f is delta differentiable at t if and only if the limit

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case,

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

(d) If f is delta differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

(e) If f is continuous at a left-scattered point t, then f is nabla differentiable at t with

$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

(f) If t is left-dense, then f is nabla differentiable at t if and only if the limit

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case,

$$f^\nabla(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

(g) If f is nabla differentiable at t, then

$$f(\rho(t)) = f(t) - \nu(t) f^\nabla(t).$$

Hereafter $[a,b]_{\mathbb{T}}$ represents an interval on time scale $\mathbb{T}$, that is, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$. Other types of intervals on a time scale can be represented by the similar way.

Lemma 2.14 *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function, then*

(a) If $f^{\Delta} > 0$ on $[a,b]_{\mathbb{T}}$, then f is strictly increasing on $[a,b]_{\mathbb{T}}$.

(b) If $f > 0$ is a continuous function on $[a,b]_{\mathbb{T}}$, then $\int_a^b f(t)\Delta t > 0$ and $\int_a^b f(t)\nabla t > 0$, where $a,\ b \in \mathbb{T}$.

Lemma 2.15 *Assume that $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable and f^{∇} is continuous on $\mathbb{T}_{\mathcal{K}}$. Then f is delta differentiable at t and*

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)) \qquad for\ all\ t \in \mathbb{T}^{\mathcal{K}}.$$

3 Maximum principles for the elliptic dynamic operators

In this section, we first consider the dynamic Laplace operator

$$\Delta_{\mathbb{T}} u := \sum_{i=1}^{n} u^{\nabla_i \Delta_i}.$$

Let

$$\Lambda = [\rho_1(a_1), \sigma_1(b_1)]_{\mathbb{T}_1} \times \cdots \times [\rho_n(a_n), \sigma_n(b_n)]_{\mathbb{T}_n}.$$

We shall study the functions in the set

$$\mathcal{D}(\Lambda) := \{u : \Lambda \to \mathbb{R} \mid u^{\nabla_i \Delta_i} \text{ is continuous in } \Lambda_{\mathcal{K}}^{\mathcal{K}} \text{ for each } i \in I\}.$$

The following lemma provides some basic properties for an interior maximum point of a function in $\mathcal{D}(\Lambda)$.

Lemma 3.1 *Suppose that $u \in \mathcal{D}(\Lambda)$ attains its maximum at an interior point m of Λ. Then, for each $i \in I$, we have*

$$u^{\nabla_i}(m) \geq 0, \qquad u^{\Delta_i}(m) \leq 0, \qquad \textit{and} \qquad u^{\nabla_i \Delta_i}(m) \leq 0.$$

In particular, if m is i-right dense, then

$$u^{\nabla_i}(m) = u^{\Delta_i}(m) = 0.$$

Proof. Since u attains its maximum at an interior point m of Λ, it follows from the definition of u^{∇_i} and u^{Δ_i} that

$$u^{\nabla_i}(m) \geq 0 \qquad \textit{and} \qquad u^{\Delta_i}(m) \leq 0, \tag{1}$$

for each $i \in I$. Let us divide our proof into two cases according to the point type of m with respect to the ith component.

(i) m is i-right dense:

In this case, by applying Lemma 2.15, we have that

$$u^{\Delta_i}(m) = u^{\nabla_i}(\hat{\sigma}_i(m)) = u^{\nabla_i}(m),$$

and consequently, together with (1), we conclude that

$$u^{\nabla_i}(m) = u^{\Delta_i}(m) = 0.$$

Now we want to show that $u^{\nabla_i\Delta_i}(m) \le 0$. For contradiction, we assume that $u^{\nabla_i\Delta_i}(m) > 0$. Then the continuity of $u^{\nabla_i\Delta_i}$ and Lemma 2.14 imply that there exists a $\delta > 0$ such that u^{∇_i} is strictly increasing in t_i on J, where J denotes the set of all points $t \in \Lambda$ lying on the line segment joining m and $m + \delta e_i$, where $\{e_i \mid i \in I\}$ denotes the natural basis for $\mathbb{R}^n$. Since m is i-right dense, without loss of generality, we may assume that $m_i + \delta \in \mathbb{T}_i$. Since $u^{\nabla_i}(m) = 0$, it follows that $u^{\nabla_i}(t) > 0$ for all $t \in J \setminus \{m\}$. Then, by applying Lemma 2.14, we easily get

$$\int_{m_i}^{m_i+\delta} u^{\nabla_i}(m_1, m_2, \cdots, m_{i-1}, s, m_{i+1}, \cdots, m_n)\nabla s = u(m+\delta e_i) - u(m) > 0,$$

which contradicts the fact that $u(m)$ is the maximum value on Λ.

(ii) m is i-right scattered:

Note that

$$u^{\nabla_i}(\hat{\sigma}_i(m)) = \frac{u(\hat{\sigma}_i(m)) - u(\hat{\rho}_i(\hat{\sigma}_i(m)))}{\sigma_i(m_i) - \rho_i(\sigma_i(m_i))} = \frac{u(\hat{\sigma}_i(m)) - u(m)}{\sigma_i(m_i) - m_i} = u^{\Delta_i}(m).$$

Together with (1), we obtain

$$u^{\nabla_i\Delta_i}(m) = \frac{u^{\nabla_i}(\hat{\sigma}_i(m)) - u^{\nabla_i}(m)}{\sigma_i(m_i) - m_i} = \frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\sigma_i(m_i) - m_i} \le 0.$$

□

Theorem 3.2 *If $u \in \mathcal{D}(\Lambda)$ satisfies*

$$\Delta_{\mathbb{T}} u > 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}}, \tag{2}$$

then u cannot attain its maximum at an interior point of Λ.

Proof. For contradiction, we assume that u attains its maximum at an interior point m of Λ. By applying Lemma 3.1, we have that $u^{\nabla_i\Delta_i}(m) \le 0$ for each

$i \in I$. This implies that

$$\Delta_{\mathbb{T}} u(m) = \sum_{i=1}^{n} u^{\nabla_i \Delta_i}(m) \leq 0,$$

which contradicts (2). $\square$

Next we consider the more general operator which contains the first-derivative terms

$$\mathcal{L}[u] := \sum_{i=1}^{n} (u^{\nabla_i \Delta_i} + B_i u^{\Delta_i} + C_i u^{\nabla_i}) = \Delta_{\mathbb{T}} u + \sum_{i=1}^{n} (B_i u^{\Delta_i} + C_i u^{\nabla_i}).$$

Following the statement of Lemma 3.1, for each $t \in \Lambda$, we define the auxiliary index sets

$$I_{RD}^{t} := \{i \in I : t_i = \sigma_i(t_i)\},$$
$$I_{RS}^{t} := \{i \in I : t_i < \sigma_i(t_i)\}.$$

Theorem 3.3 *If $u \in \mathcal{D}(\Lambda)$ satisfies*

$$\mathcal{L}[u] > 0, \qquad \textit{in } \Lambda_{\mathcal{K}}^{\mathcal{K}}, \tag{3}$$

and let B_i and C_i satisfy

$$\begin{cases} B_i(t) \geq 0, \\ C_i(t) \leq 0, \end{cases} \tag{4}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum at an interior point of Λ.

Proof. For contradiction, we assume that u attains its maximum at an interior point m of Λ. Lemma 3.1 yields that at the point m, we have

$$u^{\Delta_i}(m) = 0,\ u^{\nabla_i}(m) = 0, \text{ and } u^{\nabla_i \Delta_i}(m) \leq 0 \qquad \text{if } i \in I_{RD}^{m},$$
$$u^{\Delta_i}(m) \leq 0,\ u^{\nabla_i}(m) \geq 0, \text{ and } u^{\nabla_i \Delta_i}(m) \leq 0 \qquad \text{if } i \in I_{RS}^{m}.$$

Therefore, together with the assumption (4), we have that

$$
\begin{aligned}
&\mathcal{L}[u](m) \\
&= \sum_{i=1}^{n}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)) \\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)) \\
&\leq 0,
\end{aligned}
$$

which contradicts (3). □

Theorem 3.4 *Let $u \in \mathcal{D}(\Lambda)$ satisfy the inequality (3) and let B_i and C_i satisfy*

$$
\begin{cases} 1 + B_i(t)\mu_i(t_i) \geq 0, \\ -1 + C_i(t)\mu_i(t_i) \leq 0, \end{cases} \tag{5}
$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum at an interior point of Λ.

Proof. For contradiction, we assume that u attains its maximum at an interior point m of Λ. Then, by applying Lemma 3.1, we can rewrite $\mathcal{L}[u](m)$ in the following way:

$$
\begin{aligned}
&\mathcal{L}[u](m) \\
&= \sum_{i=1}^{n}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)) \\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)) \\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m}(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m_i)} + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)).
\end{aligned} \tag{6}
$$

If $I = I_{RD}^m$, then (6) implies that

$$
\mathcal{L}[u](m) = \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) \leq 0,
$$

which contradicts (3). Otherwise, let us define the auxiliary functions

$$\hat{\mu}(t) := \prod_{j \in I_{RS}^t} \mu_j(t_j), \qquad \hat{\mu}_{-i}(t) := \prod_{\substack{j \in I_{RS}^t \\ j \neq i}} \mu_j(t_j).$$

Obviously, if $i \in I_{RS}^t$ we have

$$\hat{\mu}(t) = \hat{\mu}_{-i}(t)\mu_i(t_i). \tag{7}$$

We multiply both sides of the equality (6) by $\hat{\mu}(m) > 0$ and use (7) to obtain

$$\begin{aligned}
&\hat{\mu}(m)\mathcal{L}[u](m) \\
= &\hat{\mu}(m) \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) \\
&+\hat{\mu}_{-i}(m)\mu_i(m_i) \sum_{i \in I_{RS}^m} \left(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m_i)} + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)\right) \\
= &\hat{\mu}(m) \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) \\
&+\hat{\mu}_{-i}(m) \sum_{i \in I_{RS}^m} [(1 + B_i(m)\mu_i(m_i))u^{\Delta_i}(m) + (-1 + C_i(m)\mu_i(m_i))u^{\nabla_i}(m)].
\end{aligned}$$

Lemma 3.1 together with the assumptions (5), and positivity of $\hat{\mu}(m)$ and $\hat{\mu}_{-i}(m)$ imply that

$$\hat{\mu}(m)\mathcal{L}[u](m) \leq 0,$$

which contradicts (3). Therefore we conclude that u cannot achieve its maximum at an interior point of Λ. $\square$

4 Maximum principles for the parabolic dynamic operators

In this section, we extend our results in the last section to the parabolic dynamic operators. Let Λ be an n-dimensional time scale defined in Section 3. Then we define the $(n+1)$-dimensional time scale Ω by

$$\Omega = \Lambda \times [0,T]_{\mathbb{T}_{n+1}},$$

where $\mathbb{T}_{n+1}$ is an arbitrary time scale and $0, T \in \mathbb{T}_{n+1}$. In addition, we set

$$B = \Lambda \times \{0\} \quad \text{and} \quad S = \partial\Lambda \times (0,T]_{\mathbb{T}_{n+1}},$$

then we can define the parabolic boundary $P\Omega$ by

$$P\Omega = S \cup B.$$

Throughout this section, we study the functions in the set

$$\mathcal{D}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u^{\nabla_i \Delta_i} \text{ is continuous in } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times [0,T]_{\mathbb{T}_{n+1}} \text{ for each } i \in I$$
$$\text{and } u^{\nabla_{n+1}} \text{ is continuous in } \Lambda \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}\}.$$

Theorem 4.1 *If $u \in \mathcal{D}(\Omega)$ satisfies*

$$\Delta_{\mathbb{T}} u - u^{\nabla_{n+1}} = \sum_{i=1}^{n} u^{\nabla_i \Delta_i} - u^{\nabla_{n+1}} > 0, \quad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{8}$$

Then u cannot attain its maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains its maximum at a point $m \in \Omega \setminus P\Omega$. This implies that $m \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$. Therefore, by applying Lemma 3.1, we have

$$u^{\nabla_i \Delta_i}(m) \le 0 \quad for\ each\ i \in I.$$

Since u attains its maximum at m, by the definition of partial nabla derivative of u, we obtain

$$u^{\nabla_{n+1}}(m) \geq 0. \tag{9}$$

It follows that

$$(\Delta_{\mathbb{T}} u - u^{\nabla_{n+1}})(m) = \sum_{i=1}^{n} u^{\nabla_i \Delta_i}(m) - u^{\nabla_{n+1}}(m) \leq 0,$$

which contradicts (8). $\square$

Similarly, we consider the more general operator

$$L[u] := \sum_{i=1}^{n} (u^{\nabla_i \Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}}.$$

Theorem 4.2 *If $u \in \mathcal{D}(\Omega)$ satisfies*

$$L[u] > 0, \qquad \text{in } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{10}$$

and let $\tilde{B}_i$ and $\tilde{C}_i$ satisfy

$$\begin{cases} \tilde{B}_i(t) \geq 0, \\ \tilde{C}_i(t) \leq 0, \end{cases} \tag{11}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains its maximum at a point $m \in \Omega \setminus P\Omega$. Lemma 3.1 together with the assumptions (11) and (9) imply that

$$\begin{aligned}
& L[u](m) \\
&= \sum_{i=1}^{n} (u^{\nabla_i \Delta_i}(m) + \tilde{B}_i(m) u^{\Delta_i}(m) + \tilde{C}_i(m) u^{\nabla_i}(m)) - u^{\nabla_{n+1}}(m) \\
&= \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) + \sum_{i \in I_{RS}^m} (u^{\nabla_i \Delta_i}(m) + \tilde{B}_i(m) u^{\Delta_i}(m) + \tilde{C}_i(m) u^{\nabla_i}(m)) - u^{\nabla_{n+1}}(m) \\
&\leq 0,
\end{aligned}$$

which contradicts (10). $\square$

Theorem 4.3 *Let $u \in \mathcal{D}(\Omega)$ satisfy the inequality (10) and let $\tilde{B}_i$ and $\tilde{C}_i$ satisfy*

$$\begin{cases} 1 + \tilde{B}_i(t)\mu_i(t_i) \geq 0, \\ -1 + \tilde{C}_i(t)\mu_i(t_i) \leq 0, \end{cases} \tag{12}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains its maximum at a point $m \in \Omega \setminus P\Omega$. As similar as the proof of Theorem 3.4, we rewrite $L[u](m)$ in the following way:

$$\begin{aligned} & L[u](m) \\ = & \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) \\ & + \sum_{i \in I_{RS}^m} \left(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m_i)} + \tilde{B}_i(m)u^{\Delta_i}(m) + \tilde{C}_i(m)u^{\nabla_i}(m)\right) - u^{\nabla_{n+1}}(m). \end{aligned} \tag{13}$$

If $I = I_{RD}^m$, then (13) and (9) imply that

$$L[u](m) = \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) - u^{\nabla_{n+1}}(m) \leq 0,$$

which contradicts (10). Otherwise, we multiply both sides of the equality (13) by $\hat{\mu}(m) > 0$ and use (7) and (9) to obtain that

$$\begin{aligned}
&\hat{\mu}(m)L[u](m) \\
&= \hat{\mu}(m)\sum_{i\in I^m_{RD}} u^{\nabla_i\Delta_i}(m) \\
&\quad +\hat{\mu}_{-i}(m)\sum_{i\in I^m_{RS}} [(1+\tilde{B}_i(m)\mu_i(m_i))u^{\Delta_i}(m)+(-1+\tilde{C}_i(m)\mu_i(m_i))u^{\nabla_i}(m)] \\
&\quad -\hat{\mu}(m)u^{\nabla_{n+1}}(m) \\
&\le 0,
\end{aligned}$$

which contradicts (10) and the proof is done. □

Next we consider the operator which contains the non-derivative term

$$(L+h)[u] := \sum_{i=1}^{n}(u^{\nabla_i\Delta_i}+\tilde{B}_i u^{\Delta_i}+\tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}} + hu.$$

Theorem 4.4 *Let $u \in \mathcal{D}(\Omega)$ satisfy*

$$(L+h)[u] > 0, \qquad in\ \Lambda^{\mathcal{K}}_{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{14}$$

and let $\tilde{B}_i$ and $\tilde{C}_i$ satisfy the inequality (12). Moreover, we suppose that

$$h(t) \le 0, \tag{15}$$

for each $t \in \Lambda^{\mathcal{K}}_{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$. Then u cannot attain a nonnegative maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains a nonnegative maximum at a point $m \in \Omega \setminus P\Omega$. By the proof of Theorem 4.3, we know that

$$L[u](m) \le 0,$$

if u attains its maximum at the point m. Then, together with the condition $h(m)u(m) \le 0$, we easily see that

$$(L+h)[u](m) = L[u](m) + h(m)u(m) \le 0,$$

which contradicts (14). □

Theorem 4.5 *If $u \in \mathcal{D}(\Omega)$ satisfies*

$$\sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i} + \beta_i u^{\hat{\sigma}_i} + \gamma_i u^{\hat{\rho}_i}) - u^{\nabla_{n+1}} + hu > 0, \tag{16}$$

in $\Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$. Further, we assume that

$$\begin{cases} 1 + (\tilde{B}_i(t) + \mu_i(t_i)\beta_i(t))\mu_i(t_i) \geq 0, \\ -1 + (\tilde{C}_i(t) - \nu_i(t_i)\gamma_i(t))\mu_i(t_i) \leq 0, \end{cases} \tag{17}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ which is i-right scattered and $i \in I$, and

$$h + \sum_{i=1}^{n}(\beta_i + \gamma_i) \leq 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}. \tag{18}$$

Then u cannot attain a nonnegative maximum anywhere other than on the parabolic boundary.

Proof. Using the formulas (d) and (g) in the Lemma 2.13, we can obtain the two analogues equalities:

$$u(\hat{\sigma}_i(t)) = u(t) + \mu_i(t_i)u^{\Delta_i}(t),$$
$$u(\hat{\rho}_i(t)) = u(t) - \nu_i(t_i)u^{\nabla_i}(t),$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ and $i \in I$. Substituting these into (16), we obtain

$$\sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + (\tilde{B}_i + \mu_i(t_i)\beta_i)u^{\Delta_i} + (\tilde{C}_i - \nu_i(t_i)\gamma_i)u^{\nabla_i}) - u^{\nabla_{n+1}} + (h + \sum_{i=1}^{n}(\beta_i + \gamma_i))u > 0.$$

Obviously, this operator has the form of (14), and the assumptions (17) and (18) ensure that the inequalities (12) and (15) hold. Consequently, we can use Theorem 4.4 to verify the statement. □

Finally, we establish the weak maximum principles for the parabolic dynamic operators and apply it to obtain the uniqueness of solutions for the initial boundary value problem.

Theorem 4.6 *Let $u \in \mathcal{D}(\Omega)$ satisfy*

$$L[u] \geq 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{19}$$

and we assume that $\tilde{B}_i$ be bounded above and $\tilde{C}_i \leq 0$ satisfy the inequalities (12). Then u attains its maximum on the parabolic boundary, i.e.,

$$\sup_{\Omega} u = \sup_{P\Omega} u. \tag{20}$$

Proof. Since $\tilde{B}_1$ is bounded above, there exists a negative constant α such that

$$\alpha + \tilde{B}_1 < 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}. \tag{21}$$

Select any point $\hat{t} \in \mathbb{T}_1$. Then, applying Lemma 2.12 and 2.15, we obtain

$$\begin{aligned} L[\hat{e}_\alpha(t_1,\hat{t})] &= (\hat{e}_\alpha(t_1,\hat{t}))^{\nabla_1\Delta_1} + \tilde{B}_1(\hat{e}_\alpha(t_1,\hat{t}))^{\Delta_1} + \tilde{C}_1(\hat{e}_\alpha(t_1,\hat{t}))^{\nabla_1} \\ &= (\alpha + \tilde{B}_1)\hat{e}_\alpha^{\Delta_1}(t_1,\hat{t}) + \alpha\tilde{C}_1\hat{e}_\alpha(t_1,\hat{t}) \\ &= (\alpha + \tilde{B}_1)\hat{e}_\alpha^{\nabla_1}(\sigma_1(t_1),\hat{t}) + \alpha\tilde{C}_1\hat{e}_\alpha(t_1,\hat{t}) \\ &= (\alpha + \tilde{B}_1)\alpha\hat{e}_\alpha(\sigma_1(t_1),\hat{t}) + \alpha\tilde{C}_1\hat{e}_\alpha(t_1,\sigma_1(t_1))\hat{e}_\alpha(\sigma_1(t_1),\hat{t}) \\ &= \alpha\hat{e}_\alpha(\sigma_1(t_1),\hat{t})[\alpha + \tilde{B}_1 + \tilde{C}_1\hat{e}_\alpha(t_1,\sigma_1(t_1))]. \end{aligned} \tag{22}$$

The assumption $\tilde{C}_1 \leq 0$ together with (21), we see that

$$L[\hat{e}_\alpha(t_1,\hat{t})] > 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}.$$

Then for each $\varepsilon > 0$, we have

$$L[u + \varepsilon\hat{e}_\alpha(t_1,\hat{t})] = L[u] + \varepsilon L[\hat{e}_\alpha(t_1,\hat{t})] > 0, \tag{23}$$

in $\Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$, so that

$$\sup_{\Omega}(u + \varepsilon\hat{e}_\alpha(t_1,\hat{t})) = \sup_{P\Omega}(u + \varepsilon\hat{e}_\alpha(t_1,\hat{t})), \tag{24}$$

by applying the Theorem 4.3.

Now we want to show that $\sup_{\Omega} u = \sup_{P\Omega} u$. For contradiction, we assume that $\sup_{\Omega} u > \sup_{P\Omega} u$. Since the time scale $\mathbb{T}_1$ is bounded, this implies that $0 < \hat{e}_\alpha(t_1, \hat{t}) < M$ for some $M > 0$. We set $K = \sup_{\Omega} u - \sup_{P\Omega} u > 0$ and take $\varepsilon = \frac{K}{2M}$, then by applying (24) we can deduce that

$$\begin{aligned}\sup_{P\Omega}(u + \varepsilon \hat{e}_\alpha(t_1, \hat{t})) &\leq \sup_{P\Omega}(u + \varepsilon M) = \sup_{P\Omega} u + \varepsilon M \\ &= (\sup_{\Omega} u - K) + \frac{K}{2} < \sup_{\Omega} u \\ &\leq \sup_{\Omega}(u + \varepsilon \hat{e}_\alpha(t_1, \hat{t})) = \sup_{P\Omega}(u + \varepsilon \hat{e}_\alpha(t_1, \hat{t})),\end{aligned}$$

which is a contradiction and the proof is done. □

The above proven maximum principles yields the uniqueness of solutions for the following problem:

$$\begin{cases} \sum_{i=1}^{n}(u^{\nabla_i \Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}} = f(t) \quad \text{on } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \\ u(t) = g(t) \quad \text{on } B, \\ u(t) = h(t) \quad \text{on } S. \end{cases} \tag{25}$$

Theorem 4.7 *Suppose that the assumptions of Theorem 4.6 hold. If u_1 and u_2 are solutions of the initial boundary value problem (25), then $u_1 \equiv u_2$.*

Proof. First of all, we define the auxiliary function $v = u_1 - u_2$. Since both u_1 and u_2 are solutions of (25), this implies that

$$\begin{cases} \sum_{i=1}^{n}(v^{\nabla_i \Delta_i} + \tilde{B}_i v^{\Delta_i} + \tilde{C}_i v^{\nabla_i}) - v^{\nabla_{n+1}} = 0 \quad \text{on } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \\ v(t) = 0 \quad \text{on } P\Omega \end{cases} \tag{26}$$

Obviously, we know that $-v$ is also a solution of (26). Then by applying Theorem 4.6, we have that

$$\sup_{\Omega} v = \sup_{P\Omega} v = 0 \qquad \textit{and} \qquad \sup_{\Omega}(-v) = \sup_{P\Omega}(-v) = 0.$$

It follows that

$$v(t) \leq 0 \qquad \textit{and} \qquad -v(t) \leq 0,$$

for each $t \in \Omega$. Consequently, we get the conclusion that $v = u_1 - u_2 = 0$. $\square$

References

[1] M. Protter, H. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, New Jersey, (1967).

[2] H. Kuo, N. Trudinger, On the discrete maximum principle for parabolic difference operators, Math. Model. Numer. Anal. 27 (1993) 719-737.

[3] G. David, N.S. Trudinger, Elliptic partial differential equations of second order, Berlin, New York: Springer-Verlag, (1977).

[4] P. Stehlik, B. Thompson, Maximum principles for second order dynamic equations on time scales, J. Math. Anal. Appl. 331 (2007) 913-926.

[5] P. Stehlik, Maximum principles for elliptic dynamic equations, Mathematical and Computer Modelling 51 (2010) 1193-1201.

[6] R.P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, Results Math. 35 (1999) 3-22.

[7] M. Bohner and A. Peterson, Dynamic Equation on Time Scales, An Introduction with Application, Birkhauser, Boston (2001).

[8] M. Bohner and A. Peterson, Advances in Dynamic Equation on Time Scales, Birkhauser, Boston (2003).

[9] B. Jackson, Partial dynamic equations on time scales, J. Comput. Appl. Math. 186 (2006) 391-415.