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1 Introduction

Different economic or econometric theories usually suggest non-nested models in theoretical and empirical researches. Tests for non-nested hypotheses, henceforth the non-nested tests, are important because researchers are able to choose the true model from non-nested models by the tests. The pioneering works of Cox (1961, 1962), Atkinson (1970) and Pesaran and Deaton (1978) become available to comparing non-nested models. Several papers, such as Davidson and MacKinnon (1981), Fisher and McAleer (1981), Gourieroux, Monfort and Trognon (1983), Mizon and Richard (1986) and Vuong (1989), discuss the theoretical methods for non-nested tests. Many papers apply the non-nested tests in empirical applications; complete surveys can be found in Gourieroux and Monfort (1994) and McAleer (1995). It is often the case that empirical fact displays non-Gaussian behavior, such as models with heavy tail or influential outliers. Most of the existing testing procedures are designed for model with Gaussian (normal) distribution and are not robust with respect to misspecification of error distribution.

Aguirre-Torres and Gallant (1983) and Hall (1985) have suggested non-nested tests that incorporate M-estimators and base on classical testing procedure. Although M-estimator is a robust estimator in general, their tests using classical procedure are lack of robustness for model where the error distribution is assumed non-Gaussian. To the best of our knowledge, only Victoria-Feser (1997) constructs a robust non-nested test. She considers a Lagrange multiplier version of the Cox test and extends the optimal bounded influence parametric tests of Heritier and Ronchetti (1994) for testing non-nested hypotheses. Her test limits the influence of small contamination in the data and is robust to model deviations. In order to derive her test statistic, one must specify an explicit density function under the null hypothesis to obtain the log-likelihood function and maximum likelihood (ML) estimators of the model. The test is thus restrictive and strong when applying in practice. In addition, similar to the Cox test, the test statistic involves a very difficult integration problem such as the Cox test and are not easy to compute for applied theorists.

In this paper, we propose a robust testing procedure for the non-nested hypotheses. Several features are as follows. First, the proposed test extends the rank score test of Gutenbrunner *et al.* (1993); this class of rank test plays an important role especially when the empirical phenomenons are non-Gaussian. Second, the rank test statistic is based on the regression rankscore process that is computed from parametric linear programming method of quantile regression. Specifications of explicit density functions are not required. Also, we do not need to estimate ML estimators and only root-n consistent estimators of non-nested models can be used in the proposed test. Different from the test of Victorian-Feser (1997), it is easy to compute our statistics. In addition, the propsed test is easy to implement by existing software. Unlike the non-nested test in general use, the simulation or bootstrapping methods are not required. Third, we show that under very weak assumptions, the proposed test statistic has asymptotically χ^2 distribution and the test is asymptotic distribution-free. Fourth, the proposed test can be extended to test one model against several competing models. The choice of multiple model selection becomes available. Fifth, local powers of our robust rank score test are derived. Finally, Monte Carlo simulations results are provided and show that the proposed test has good finite sample performances. Comparing with the J test, the rank score tests is robust when the error term is non-Gaussian. Moreover, unlike the existing non-nested literature, our test is robust to the relative number of regressors in the two hypotheses.

This paper is organized as follows. In section 2, the rank score tests for non-nested hypotheses are proposed. We consider both single and multiple alternatives. Local alternatives of our test are discussed in section 3. Some Monte Carlo simulation results are presented in section 4. Section 5 is our conclusion of this paper.

2 Rank Score Tests

2.1 Motivations and Setup

Suppose that we want to choose between two linear models as follows:

$$H_0 : \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}_0,$$

$$H_1 : \mathbf{y} = \mathbf{Z}\boldsymbol{\gamma} + \mathbf{e}_1,$$

where the dependent variable $\mathbf{y}$ is an $n \times 1$ matrix, explanatory variables $\mathbf{X}$ and $\mathbf{Z}$ are $n \times p$ and $n \times q$ matrices, and $\mathbf{e}_0$ and $\mathbf{e}_1$ are error terms, respectively. $\mathbf{X}$ and $\mathbf{Z}$ are two matrices which contain different variables and the models of H_0 and H_1 are non-nested. To test non-nested hypotheses H_0 and H_1, we consider the following artificial nesting model:

$$\mathbf{y} = (1-\lambda)\mathbf{X}\boldsymbol{\beta} + \lambda\mathbf{Z}\boldsymbol{\gamma} + \mathbf{e}, \tag{1}$$

where $\lambda = 0$ means that the null hypothesis is correct and $\lambda = 1$ means that the alternative hypothesis is correct. Under this artificial nesting model, we are able to reconsider the non-nested hypotheses as

$$H_0 : \lambda = 0,$$

$$H_1 : \lambda = 1.$$

We can test the non-nested hypotheses by testing $\lambda = 0$ against $\lambda = 1$ for nesting model (1).

Davidson and MacKinnon (1981) estimated $\boldsymbol{\gamma}$ in (1) by its ML estimator $\tilde{\boldsymbol{\gamma}}$ and then estimated λ from model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \lambda\mathbf{Z}\tilde{\boldsymbol{\gamma}} + \mathbf{e}, \tag{2}$$

with error density $\mathbf{e}$ independently and identically distributed (i.i.d.) as normal distributions. Davidson and MacKinnon introduced the J test that uses the classical t statistic for $\hat{\lambda}$ to test the non-nested hypotheses $\lambda = 0$. The J test is based on the classical testing procedure and is not robust to the misspecification of error density. For example, if the error distributions of the non-nested models are non-Gaussian, the J test may lead to incorrect inference.

To overcome the non-robustness problem discussed above, Victoria-Feser (1997) proposes a robust test using the optimal bounded influence parametric tests of Heritier and Ronchetti (1994). The test is to limit the influence of small contamination in the data. Victoria-Feser (1997) considers a Lagrange multiplier version of the Cox test and bounds the level influence function of the test. As one can see that her test bounds the effect of the outlier and is a first paper for robust test for non-nested hypotheses. It is however, in the context of ML method, complete density functions of the models should be specified. In addition, one needs to compute the ML estimator in her test. This makes her test statistic very complicated to compute (see p.722-723 for the computation of her test statistic). Her test is thus restrictive and not operational in practice.

2.2 A Robust Test

In this article, a robust testing procedure for non-nested hypotheses is proposed. The proposed test is based on the rank score test of Gutenbrunner *et al.* (1993), that tests the parametric hypothesis for quantile regression. For testing non-nested models in H_0 and H_1, we now apply the rank score test to test $\lambda = 0$ in the artificial nesting model (2). When $\lambda = 0$, the restricted model becomes $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$, and a regression rankscore process $\hat{\mathbf{a}}(t)$, introduced by Gutenbrunner and Jurečková (1992) is obtained by solving from

$$\hat{\mathbf{a}}(t) = \arg\max \left\{ y'\mathbf{a} \middle| \mathbf{X}'\mathbf{a} = (1-t)\mathbf{X}'\mathbf{1}_n, \mathbf{a} \in [0,1]^n \right\}, \tag{3}$$

with $\mathbf{1}_n$ an $n \times 1$ vector of ones. It is noted that problem (3) is the dual problem of the objective function of quantile regression in linear programming. $\hat{\mathbf{a}}(t)$ can be obtained easily from the existing software since the quantile regression has been available in the standard toolbox of researcher's desk.

Let $\hat{a}_i(t)$ be the i'th element of the regression rankscore process. Let $\varphi(t)$ be a score function with bounded variation. Integrating $\varphi(t)$ with respect to $\hat{a}_i(t)$ from zero to one constructs

$$\hat{b}_i = -\int_0^1 \varphi(t) d\hat{a}_i(t). \tag{4}$$

Denote $\hat{\mathbf{b}}$ with ith element $\hat{b}_i$ to be a "score" vector and contains $\hat{b}_i, i = 1, \cdots, n$. $\varphi(t)$ is also called a score generating function that generates score $\hat{b}_i$. The underlying idea of the rank score test is to check whether scores $\hat{\mathbf{b}}$ are sufficiently close to zero. Intuitively, $\hat{\mathbf{b}}$ can be interpreted as functions of ranks of residualss from restricted regression model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \mathbf{e}$. Of course, different score-generating functions $\varphi(\cdot)$ lead to different $\hat{\mathbf{b}}$. Three commonly used score functions are Wilcoxon scores, normal scores and sign-median scores. We compare the power performances of our test by different score generating functions in Section 4.

Extending the test of Gutenbrunner *et al.* (1993), we propose a rank score test for Davidson and Mackinnon's artificial nesting model (1) and define

$$\mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) = \frac{1}{\sqrt{n}}(\mathbf{Z}\hat{\boldsymbol{\gamma}} - \tilde{\mathbf{Z}})'\hat{\mathbf{b}},$$

where $\hat{\boldsymbol{\gamma}}$ can be any consistent estimator of the restricted model, and $\tilde{\mathbf{Z}}$ is the linear projection of $\mathbf{Z}\hat{\boldsymbol{\gamma}}$ on $\mathbf{X}$:

$$\tilde{\mathbf{Z}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\hat{\boldsymbol{\gamma}}.$$

It follows that

$$\mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) = \frac{1}{\sqrt{n}}(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}})'\hat{\mathbf{b}},$$

where $\mathbf{M}_X = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. The rank score test for non-nested hypotheses is defined as:

$$\mathcal{R} := \mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})'\hat{\mathbf{V}}^{-1}\mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})/\mathbf{A}^2(\varphi),$$

where $\hat{\mathbf{V}} = n^{-1}(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}})'(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}})$, and

$$\mathbf{A}^2(\varphi) = \int_0^1 \left(\varphi(t) - \int_0^1 \varphi(t)dt\right)^2 dt, \tag{5}$$

for some score function $\varphi(\cdot)$. The proposed test statistic is only composed of data $\mathbf{X}$ and $\mathbf{Z}$, an estimator $\hat{\boldsymbol{\gamma}}$, and $\hat{\mathbf{b}}$, and is easy to computed. In our test, we do not need to specify the complete density function and the estimating of ML estimator is not required. The proposed test is thus easy to implement. In the following, we show that the limiting distribution of the proposed test is chi-square distribution with one degree of freedom.

Let $\mathbb{X} = [\mathbf{X}\ \tilde{\mathbf{Z}}]$ be an $n \times (p+1)$ matrix and $\{x_i, i = 1, \cdots, n\}$ the i-th vector of $\mathbb{X}$. Denote $e_i, i = 1, \cdots, n$ to be the ith element of error vector $\mathbf{e}$ and the conditional distribution functions of error term e_i conditional on information set $\mathcal{F}$ are denoted as $F_{e_i|\mathcal{F}}(\cdot), i = 1, \cdots, n$. In addition, denote $\boldsymbol{\gamma}_{\boldsymbol{\beta}}$ as the pseudo-true estimator which is the limiting behavior of $\hat{\boldsymbol{\gamma}}$ under the model in H_0.

Theorem 2.1. *If* (i) $F_{e_i|\mathcal{F}}(\cdot), i = 1, \cdots, n$ *are i.i.d. and absolutely continuous with continuous densities* $f_{e_i|\mathcal{F}}(\cdot)$ *uniformly bounded away from 0 and* ∞*.* (ii) (a) $x_1 = \mathbf{1}_n$*, with* $\mathbf{1}_n$ *an* $n \times 1$ *vector of ones,* (b) $n^{-1}\mathbb{X}'\mathbb{X} \to \mathbf{D}$*, a positive definite matrix,* (c) $n^{-1}\sum_{i=1}^{n}||x_i||^4 = O(1)$, (d) $\max_{i=1,\cdots,n}||x_i|| = O(n^{1/4}/\log n)$. (iii) $\hat{\boldsymbol{\gamma}}$ *is a consistent estimator of* $\boldsymbol{\gamma}_{\boldsymbol{\beta}}$*.* (iv) $\hat{\mathbf{V}} \to \mathbf{V} := \mathbb{E}_0[(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})]/n$*, a positive definite matrix. Under the null hypothesis,*

$$\mathcal{R} \Rightarrow \chi_1^2.$$

Proof. Under the assumptions (i) and (ii), by the same arguments in Theorems 4.1 and 5.1 of Gutenbrunner *et al.* (1993), we have

$$\mathbf{S}\big(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}\big) = \frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}}\big)'\hat{\mathbf{b}} = \frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}}\big)'\mathbf{b} + o_p(1),$$

where

$$\mathbf{b} = -\int_0^1 \varphi(t)d(\tau - \mathbf{1}_{\{y_i - \mathbf{x}_i'\boldsymbol{\beta}_\tau < 0\}}).$$

In addition, rewrite

$$\begin{aligned}\frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}}\big)'\mathbf{b} &= \frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\big)'\mathbf{b} + \frac{1}{\sqrt{n}}\big[\mathbf{M}_X\mathbf{Z}\big(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_{\boldsymbol{\beta}}\big)\big]'\mathbf{b} \\ &= \frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\big)'\mathbf{b} + \big(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_{\boldsymbol{\beta}}\big)\frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\big)'\mathbf{b} \\ &= \frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\big)'\mathbf{b} + o_p(1),\end{aligned}$$

where the last equality holds because $\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_{\boldsymbol{\beta}} = o_p(1)$ and $n^{-1/2}\big(\mathbf{M}_X\mathbf{Z}\big)'\mathbf{b} = O_p(1)$. By central limit theorem and under the null hypothesis,

$$\frac{1}{\sqrt{n}}\big(\mathbf{M}_X\mathbf{Z}\hat{\boldsymbol{\gamma}}\big)'\mathbf{b} \Rightarrow N\big(0, \mathbf{VA}^2(\varphi)\big).$$

Therefore, under assumption (iii), one has

$$\mathcal{R} \Rightarrow \chi_1^2.$$

□

2.3 Multiple Alternatives

The testing procedure introduced in the aforementioned can be extended to the choice of multiple alternatives. Suppose that there are k different non-nested alternatives as follows:

$$H_1^1 : \mathbf{y} = \mathbf{Z}^1\boldsymbol{\gamma}^1 + \mathbf{e}_1,$$

$$\cdots$$

$$H_1^k : \mathbf{y} = \mathbf{Z}^k\boldsymbol{\gamma}^k + \mathbf{e}_k,$$

where $\mathbf{Z}^1, \cdots, \mathbf{Z}^k$ are $n \times q^1, \cdots, n \times q^k$ matrices, $\boldsymbol{\gamma}^1, \cdots, \boldsymbol{\gamma}^k$ are associated parameters, respectively, and $\mathbf{e}_1, \cdots, \mathbf{e}_k$ represent error terms. To test H_0 against multiple alternatives $H_1^1, \cdots, H_1^k$, we combine these non-nested hypotheses into an artificial nesting model as Davidson and McKinnon (1981) and McAleer (1983):

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \mathbf{W}\boldsymbol{\lambda} + \mathbf{e},$$

where $\mathbf{W} = \left(\mathbf{Z}^1\hat{\boldsymbol{\gamma}}^1, \cdots, \mathbf{Z}^k\hat{\boldsymbol{\gamma}}^k\right)$, $\hat{\boldsymbol{\gamma}}^1, \hat{\boldsymbol{\gamma}}^2, \cdots, \hat{\boldsymbol{\gamma}}^k$ are consistent estimators of $\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \cdots, \boldsymbol{\gamma}^k$, respectively, and $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k)$ contains k elements, $\mathbf{e} = (e_1, \cdots, e_n)$ is an error vector in this model. We thus can test the multiple non-nested hypotheses by testing whether all elements of $\boldsymbol{\lambda}$ significantly differ from zero or not: $H_0 : \boldsymbol{\lambda} = \mathbf{0}$.

For the multiple non-nested alternative case, the test statistic is defined as

$$\mathcal{R}_k := \mathbf{S}_k'\hat{\mathbf{V}}_k^{-1}\mathbf{S}_k/\mathbf{A}^2(\varphi),$$

where

$$\mathbf{S}_k = \frac{1}{\sqrt{n}}(\mathbf{M}_X\mathbf{W})'\hat{\mathbf{b}},$$

with $\hat{b}_i$ the ith element of $\hat{\mathbf{b}}$, $\hat{\mathbf{V}}_k = n^{-1}(\mathbf{M}_X\mathbf{W})'(\mathbf{M}_X\mathbf{W})$, and $\hat{b}_i$ and $\mathbf{A}^2(\varphi)$ are the same as defined in (4) and (5). We have the following theorem.

Corollary 2.2. *If* (i) $F_{e_i|\mathcal{F}}, i = 1, \cdots, n$ *are i.i.d. and absolutely continuous with continuous densities* f_{e_i} *uniformly bounded away from 0 and* ∞. (ii) (a) $x_1^* = \mathbf{1}_n$, *with* $\mathbf{1}_n$ *an* $n \times 1$ *vector of ones,* (b) $n^{-1}\mathbb{X}^{*\prime}\mathbb{X}^* \to \mathbf{D}^*$, *a positive definite matrix,* (c) $n^{-1}\sum_{i=1}^n ||x_i^*||^4 = O(1)$, (d) $\max_{i=1,\cdots,n} ||x_i^*|| = O(n^{1/4}/\log n)$, *and* (iii) $\hat{\mathbf{V}}_k \to \mathbf{V}_k^* := \mathbb{E}_0[(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})]/n$, *Under the null hypothesis,*

$$\mathcal{R}_k \Rightarrow \chi_k^2,$$

a chi-square distribution with k degree of freedom.

3 Local Powers of the Test

The local powers of our test are considered in this section. Pesaran (1982) and Ericsson (1983) have compared the local powers of non-nested tests. Consider a local alternative as

$$H_{1n} : \lambda_n = \frac{\lambda_0}{\sqrt{n}}.$$

Similar to (2), the resulting artificial nesting model is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \lambda_n \mathbf{Z}\hat{\boldsymbol{\gamma}} + \mathbf{e},$$

with $\mathbf{e} = (e_1, \cdots, e_n)$. As sample size n increases, the model converges to the null model.

Denote $\mathbb{E}_{1n}$ as the expectation under H_{1n}. Under H_{1n},

$$\mathbb{E}_{1n}\big[\mathbf{1}_{\{y_i - \mathbf{x}_i'\boldsymbol{\beta}_\tau < 0\}}|\mathcal{F}\big] = \mathbb{E}_{1n}\big[\mathbf{1}_{\{e_i < -\frac{\lambda_0}{\sqrt{n}}\mathbf{z}_i'\hat{\boldsymbol{\gamma}}\}}|\mathcal{F}\big] = F_{e_i|\mathcal{F}}\left(-\frac{\lambda_0}{\sqrt{n}}\mathbf{z}_i\hat{\boldsymbol{\gamma}}\right),$$

and taking Taylor expansion at zero,

$$F_{e_i|\mathcal{F}}\left(-\frac{\lambda_0}{\sqrt{n}}\mathbf{z}_i\hat{\boldsymbol{\gamma}}\right) = \tau - \left(\frac{\lambda_0}{\sqrt{n}}\mathbf{z}_i'\hat{\boldsymbol{\gamma}}\right) f_{e_i|\mathcal{F}}(0) + o_p(1).$$

It follows that

$$\begin{aligned}
\mathbb{E}_{1n}[\mathbf{b}_i|\mathcal{F}] &= \mathbb{E}_{1n}\left[-\int_0^1 \varphi(t)d\left(\tau - \mathbf{1}_{\{y_i - \mathbf{x}_i'\boldsymbol{\beta}_\tau < 0\}}\right)\big|\mathcal{F}\right] \\
&= \int_0^1 \varphi(t)d\left[\tau - \mathbb{E}_{1n}\left(\mathbf{1}_{\{y_i - \mathbf{x}_i'\boldsymbol{\beta}_\tau < 0\}}\big|\mathcal{F}\right)\right] \\
&= \left(\frac{\lambda_0}{\sqrt{n}}\mathbf{z}_i'\hat{\boldsymbol{\gamma}}\right)\int_0^1 \varphi(t)df_{e_i|\mathcal{F}}(0)
\end{aligned}$$

where the second equality holds by assuming the expectation and integration are interchangeable. If the conditional distribution are i.i.d., $f_{e_1|\mathcal{F}}(0) = f_{\mathbf{e}|\mathcal{F}}(0)$,

$$\mathbb{E}_{1n}[\mathbf{b}|\mathcal{F}] = \left(\frac{\lambda_0}{\sqrt{n}}\mathbf{Z}\hat{\boldsymbol{\gamma}}\right)\int_0^1 \varphi(t)df_{\mathbf{e}|\mathcal{F}}(0).$$

From the proof of Theorem 2.1, it is known that $\mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}) = n^{-1/2}(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'\mathbf{b} + o_p(1)$. It follows that

$$\begin{aligned}
\mathbb{E}_{1n}\left[\mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})\right] &= \mathbb{E}_{1n}\left[\frac{1}{\sqrt{n}}(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'\,\mathbb{E}_{1n}[\mathbf{b}|\mathcal{F}]\right] \\
&= \left(\frac{\lambda_0}{n}\right)\mathbb{E}\left[(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'\mathbf{Z}\hat{\boldsymbol{\gamma}}\int_0^1 \varphi(t)df_{\mathbf{e}|\mathcal{F}}(0)\right] \\
&= \left(\frac{\lambda_0}{n}\right)\omega(\varphi, f)\,\mathbb{E}_{1n}\left[(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\right],
\end{aligned}$$

with $\int_0^1 \varphi(t)df_{\mathbf{e}|\mathcal{F}}(0) = \int_0^1 \varphi(t)df_{\mathbf{e}|\mathcal{F}}(0) = \omega(\varphi, f)$, where the third equality holds by $\hat{\boldsymbol{\gamma}}$ is a consistent estimator of $\boldsymbol{\gamma}_{\boldsymbol{\beta}}$.

Therefore, under H_{1n}, $\mathbf{S}(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$ is asymptotically normal distributed with mean

$$\left(\frac{\lambda_0}{n}\right)\omega(\varphi, f)\,\mathbb{E}_{1n}\left[(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}})'\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\right]$$

and variance $\mathbf{VA}^2(\varphi)$. Under H_{1n}, $\mathcal{R}$ is noncentral χ_q^2 distribution with non-centrality parameter

$$\left(\frac{\lambda_0}{n}\right)^2 \left[\frac{\omega^2(\varphi,\ f)}{\mathbf{A}^2(\varphi)}\right] \mathbb{E}_{1n}\left[\left(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\right)'\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\right]' \mathbf{V}^{-1}\,\mathbb{E}_{1n}\left[\left(\mathbf{M}_X\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\right)'\mathbf{Z}\boldsymbol{\gamma}_{\boldsymbol{\beta}}\right].$$

4 Monte Carlo Simulations

We use Monte carlo simulations to investigate the finite sample performances and robustness of the proposed tests. The data generating process (DGP) is specified as follows. Given a weight $\omega \in [0, 1]$,

$$\mathbf{y} = (1-\omega)\mathbf{X}\boldsymbol{\beta} + \omega\mathbf{Z}\boldsymbol{\gamma} + \mathbf{e}, \tag{6}$$

where $\mathbf{X}$, $\mathbf{Z}$ are $n \times p$, $n \times q$ random matrices i.i.d. from $N(0, 1)$ except that $\mathbf{x}_1' = \mathbf{1}_n$, and $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are $p \times 1$, $q \times 1$ vector of ones. The replication of each simulation is 3000 and we compute the rejection probabilities to check the finite sample performances. When $\omega = 0$, the nesting model (6) becomes the null model of H_0; the resulting rejection probabilities are the finite sample sizes of our test. When $\omega = 1$, the nesting model becomes the alternative model of H_1 and we can obtain the finite sample power. The nominal level is 5% in this section. Four scenarios are considered: the error terms $\mathbf{e}$ are i.i.d. from standard normal distribution, normal distribution with mean 0 and variance 4, the t distribution with 2 degree of freedom, and the standard cauchy distribution.

To examine the finite sample size, we first consider that the error term is i.i.d. drawn from $N(0, 1)$ and $\omega = 0$. Different score generating functions, sign, Wilcoxon and normal, are considered in the simulation. Table 1 reports the rejection probabilities of the test with sample sizes $n = 50$ and $n = 300$. In Table 1, the finite sample size is over-sized which is common in non-nested J test; see Godfrey and Pesaran (1983) and Gourieroux and Monfort (1994). The finite sample size is more accurate when q is small and is greater than the nominal level when q is large relative p. For example, when $q = 2$, finite sample sizes from $p = 2$ to $p = 6$ are 8.3%, 6.6%, 6.2%, 6.8%, 5.6% which are close to the nominal size. Moreover, when $p = 4$, finite sample sizes from $q = 2$ to $q = 6$ are 6.2%, 7.9%, 9.4%, 11.4%, 12.7% which become greater when q is large relative to p. For different score generating functions, the finite sample performances of our test are similar but the finite sample size is smaller with sign score generating function. In addition, the finite sample size is greater when the sample size $n = 50$ than the finite sample size when $n = 300$. This result comes from the asymptotic effect.

To examine the finite sample power, we show finite sample power functions of our test in Figures 1–2 with three different score generating functions. The error term is i.i.d. drawn from standard normal distribution. The sample sizes is 300. Figure 1 is power functions for non-nested

Table 1: Finite Sample Sizes, %

	n=50					n=300				
sign										
p	q=2	q=3	q=4	q=5	q=6	q=2	q=3	q=4	q=5	q=6
2	7.0	10.7	12.5	16.1	18.8	8.3	10.7	13.1	14.0	17.9
3	7.2	9.0	10.9	12.5	16.0	6.6	7.7	10.4	12.4	14.6
4	6.3	9.3	10.3	11.0	13.8	6.2	7.9	9.4	11.4	12.7
5	7.3	8.1	9.2	9.7	12.8	6.8	6.8	9.5	11.6	10.8
6	6.4	8.0	8.1	9.2	10.4	5.6	7.9	7.4	8.9	9.9
Wilcoxon										
p	q=2	q=3	q=4	q=5	q=6	q=2	q=3	q=4	q=5	q=6
2	8.6	13.3	17.6	21.2	25.8	8.6	12.4	17.1	19.1	26.2
3	7.8	11.2	13.3	16.3	20.3	7.8	9.6	14.1	15.9	19.8
4	7.7	11.6	13.0	14.8	18.1	7.8	8.5	10.9	14.0	16.5
5	8.3	9.6	11.3	12.5	15.8	7.2	9.1	11.8	13.5	14.8
6	6.5	9.2	9.6	12.6	13.7	7.3	9.3	9.4	10.5	13.0
normal										
p	q=2	q=3	q=4	q=5	q=6	q=2	q=3	q=4	q=5	q=6
2	8.5	13.7	17.5	21.3	27.1	8.7	12.2	17.5	20.3	27.5
3	7.1	10.9	13.4	15.7	20.9	8.2	10.3	13.8	17.3	20.4
4	7.1	11.2	12.5	14.2	17.9	7.7	9.1	12.0	14.0	17.3
5	7.5	8.8	10.7	11.9	15.4	6.9	9.1	11.5	14.1	15.0
6	5.7	8.8	9.1	12.7	12.0	7.3	8.8	9.8	10.8	13.9

models with $p = q = 3$. The horizontal axis is ω and the vertical axis is the rejection probability. When ω deviates from zero, the rejection probabilities are finite sample powers. From Figure 1, we can see that our test has good power performances in all three score generating functions. When ω is about ± 0.25, the finite sample power approximates 1. The tests with Wilcoxon and normal score generating functions have almost the same power function. The test with sign score generating function has the most accurate finite sample sizes than the test with Wilcoxon and normal score generating functions. The finite sample power of the test with Wilcoxon and normal score generating functions is greater than the one of the test with sign score function. Figure 2 plots the power functions of our test for $p = 5, q = 2$ with Wilcoxon, normal and sign score generating functions. Our test has finite sample sizes which are close to the nominal size in all three score generating functions. When $\omega = \pm 0.2$, the finite sample power approximates 1 and thus the finite sample performances of the proposed test are very good. Similar to the case of $p = q = 3$, finite sample powers of the test with Wilcoxon and normal score generating functions are greater than the ones with sign functions. This result is intuitive since the error term is drawn from the normal distribution.

To examine the robustness of our test, we consider several non-standard scenarios and compare the power performances of our test and the J test. First, Figure 3 plots finite sample power functions of our test and the J test with the error term i.i.d. from $N(0, 1)$. Figures 4–5 plot finite sample power functions of our test and the J test when the error term is i.i.d. drawn from $N(0, 4)$. Both the tests are over-sized but our test has more accurate finite sample size than the one of the J test. Therefore, the over-sized problem is less serious in our test. In addition, powers of the J test are greater than powers of our test. Like Godfrey and Pesaran (1983), we adjust our test and the J test to see whether high power comes form high size. We find that our test has slightly high powers when $\omega \neq 0$. This shows that high power performances of the J test come from high size. Finally, we can see that our test and the J test both perform better when the error term is from $N(0, 1)$ than when the error term is from $N(0, 4)$.

Figures 6, 7, 8 plot finite sample power functions of our test and the J test when the error term is i.i.d. drawn from the t_2 distribution for non-nested models with $p = 2, q = 7$ and $p = q = 3$ and $p = 6, q = 3$, respectively. In Figures 6 and 7, sizes of our test are 8.2% and 8.6% which are close to the nominal size. Sizes of the J test are 17.9% and 20.8% which are much greater than the nominal size. The power of our test approximates 1 when $\omega = \pm 0.3$ and the power of the J test approximates 1 when ω are around 0.6 to ± 0.8. This shows that our test has better power performances in cases when $p = q = 3$ and $p = 6, q = 3$. The power of our test approximates 1 more rapid than the power of the J test. In addition, in Figure 8, the $p = 2, q = 7$ case, the J test performs poorly when the number of regressors in the alternative hypothesis is large relative to the one in the null hypothesis. Our test is less sensitive to the relative number of regressors in the two hypotheses. In addition, the J test is very sensitive to the relative number of regressors of the two non-nested hypotheses, while the rank score test is not. Therefore, our test is more robust than the J test.

Figures 9, 10, 11 plot finite sample power functions of our test and the J test when the error term is i.i.d. drawn from the standard Cauchy distribution for non-nested models with $p = q = 3$, $p = 6, q = 3$,and $p = 2, q = 7$. In Figures 9 and 10, sizes of our test are 6.4% and 5.6% and sizes of the J test are 28.6% and 29.0%. Our test has correct finite sample size while the finite sample size of the J test are badly distorted when the errors are not assumed to be normal. The powers of our test increase largely when ω deviates from 0 and our test has good power performances under Cauchy distribution. The powers of the J test increase very slowly when ω deviates from 0. When $\omega = \pm 1$, the powers of our test is double of the powers of the J test. In Figure 11, our test also has better power than the J test but the performance of the J test is poor. To sum up, when the DGP is non-Gaussian distribution, the small sample simulations show that our test is robust in two ways. First, our test is robust to different data generating process, especially the non-standard distribution. Second, our test is robust with respect to the number of the regressors of non-nested models.

5 Conclusion

Robust testing procedures of non-nested tests are desired in theoretical and empirical researches. Unlike the optimal bounded influence parametric test considered by Victoria-Feser (1997), we have suggested another robust test for non-nested tests by extending the regression rank score test of Gutenbrunner *et al.* (1993). We introduce a test statistic $\mathcal{R}$ for two non-nested hypotheses and the statistic $\mathcal{R}_k$ for multiple non-nested alternatives. The limiting distributions of our test statistics in this paper are χ^2 distributions and our test is asymptotically distribution free. Moreover, limiting distributions of the proposed test under the local alternative are non-central χ^2 distributions. Monte Carlo simulations show that our test has good finite sample performances against non-nested alternatives. Our test is robust for testing non-nested hypotheses under non-Gaussian error terms and is robust to the relative number of regressors in the two hypotheses.

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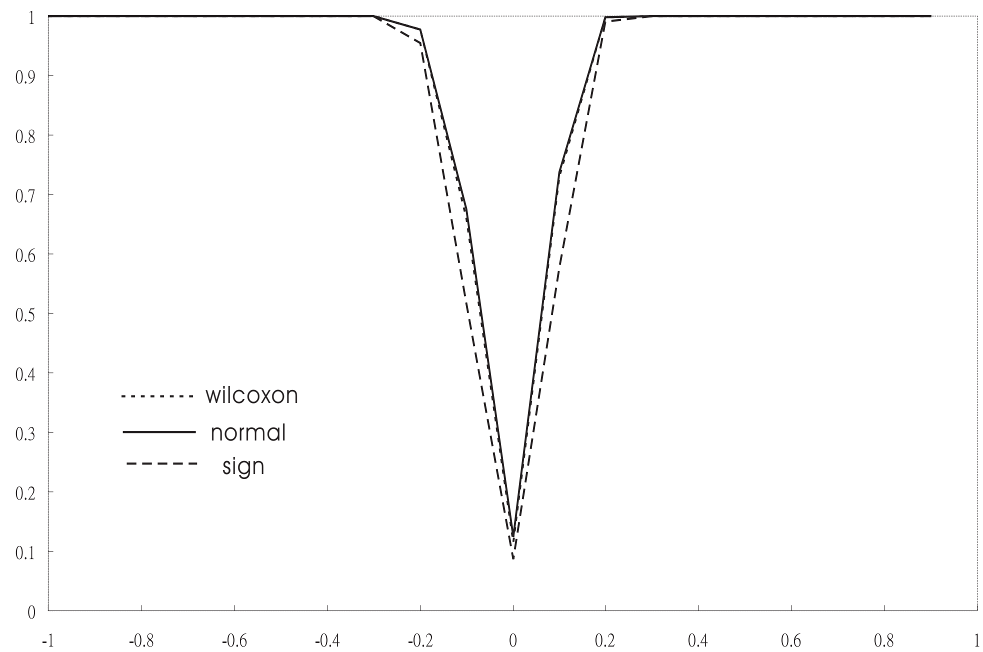

Figure 1: Finite sample powers of $\mathcal{R}$ with $p = 3$ and $q = 3$.

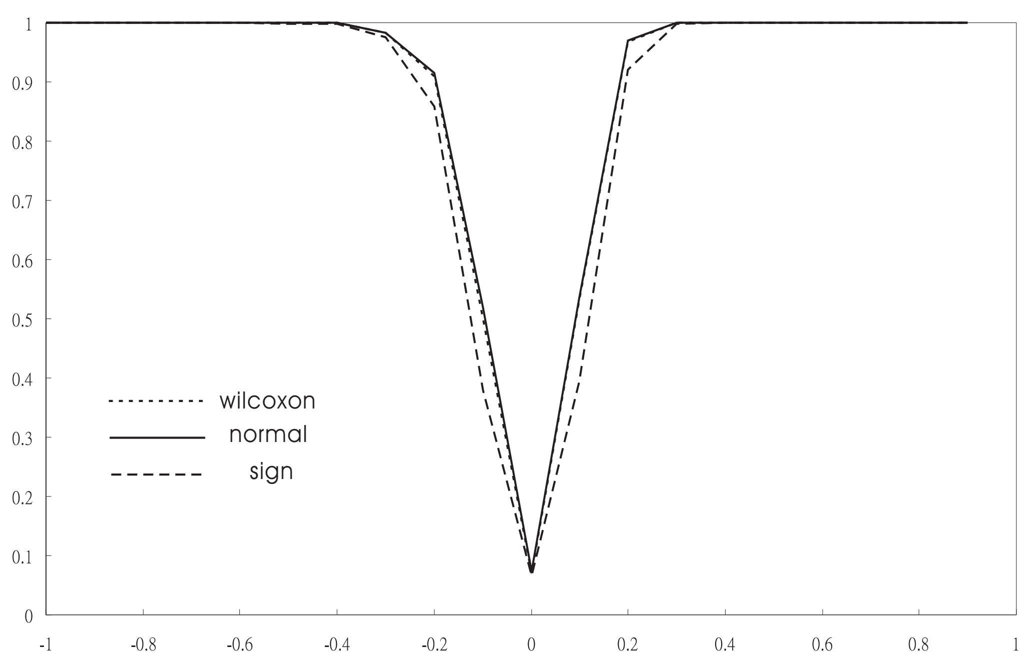

Figure 2: Finite sample powers of $\mathcal{R}$ with $p = 5$ and $q = 2$.

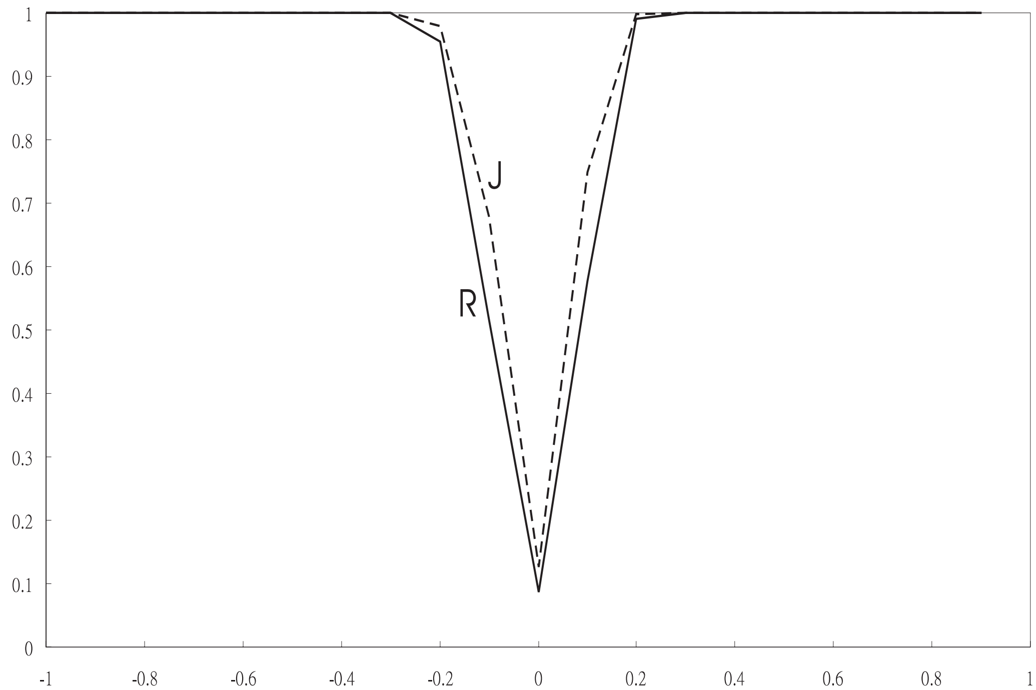

Figure 3: Finite sample powers under $N(0, 1)$ distribution and $p = 3$ and $q = 3$.

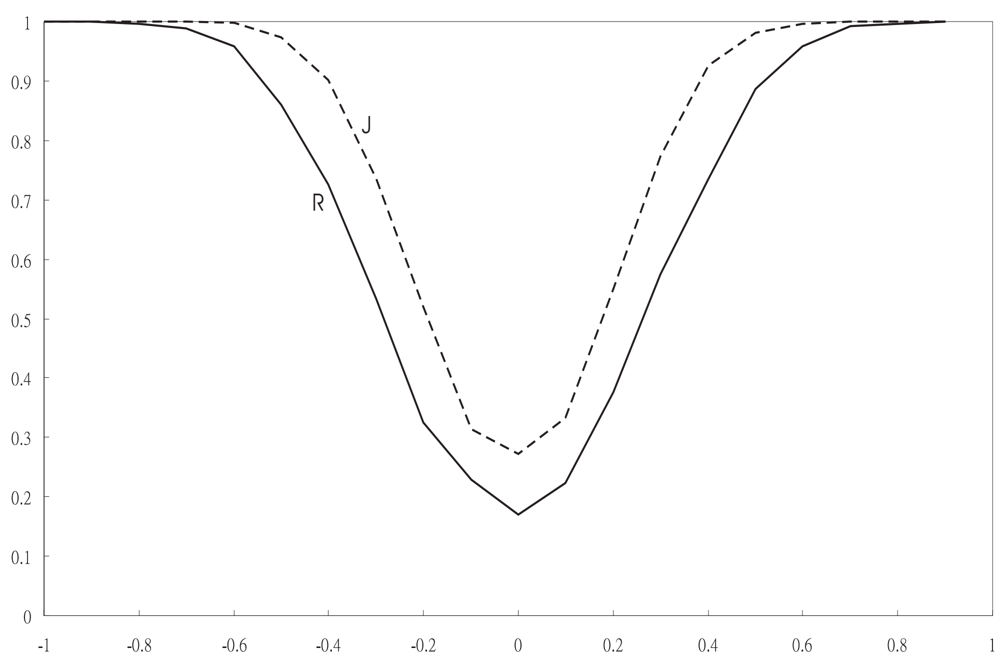

Figure 4: Finite sample powers under $N(0, 4)$ distribution and $p = 3$ and $q = 3$

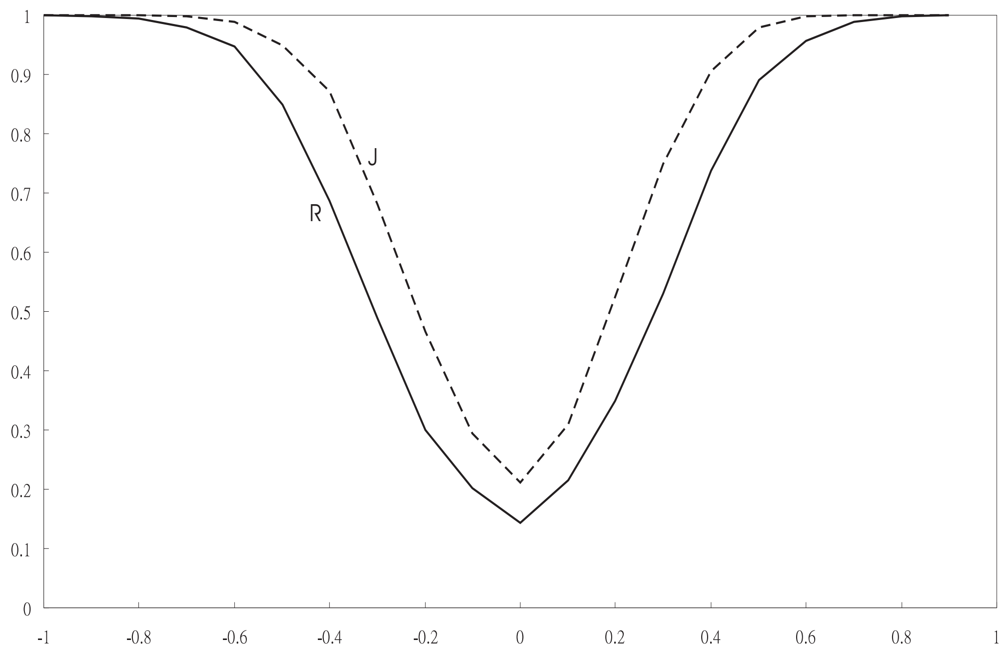

Figure 5: Finite sample powers under $N(0, 4)$ distribution and $p = 6$ and $q = 3$.

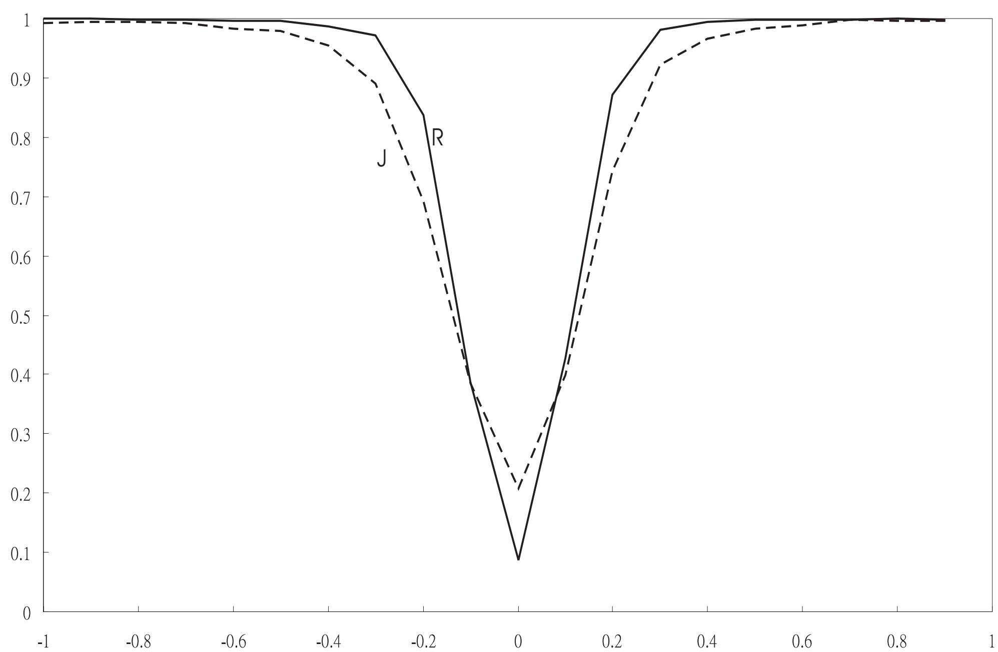

Figure 6: Finite sample powers under t_2 distribution and $p = 3$ and $q = 3$.

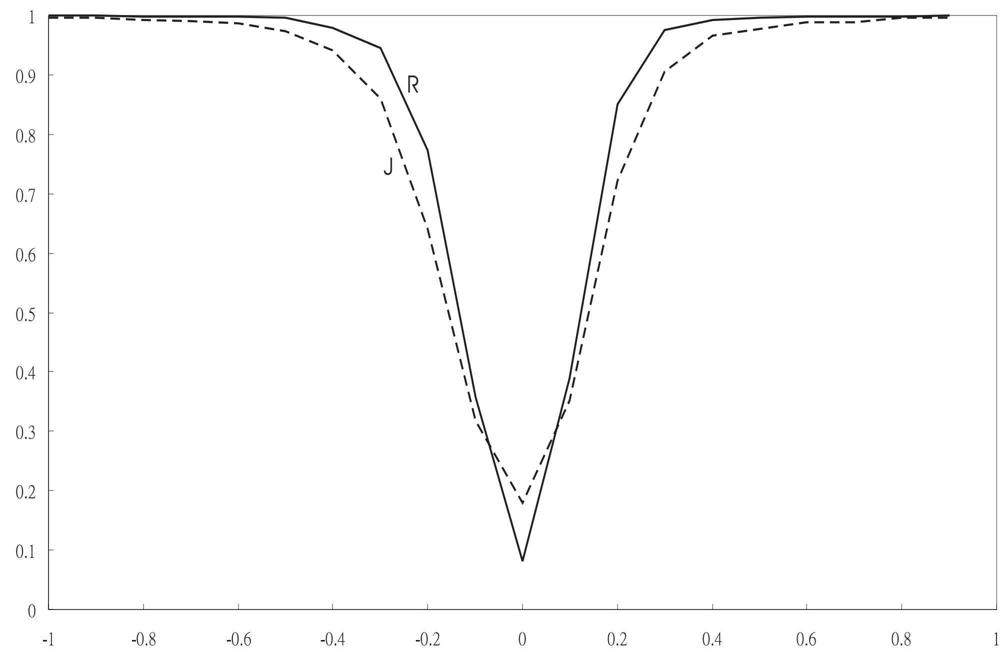

Figure 7: Finite sample powers under t_2 distribution and $p = 6$ and $q = 3$.

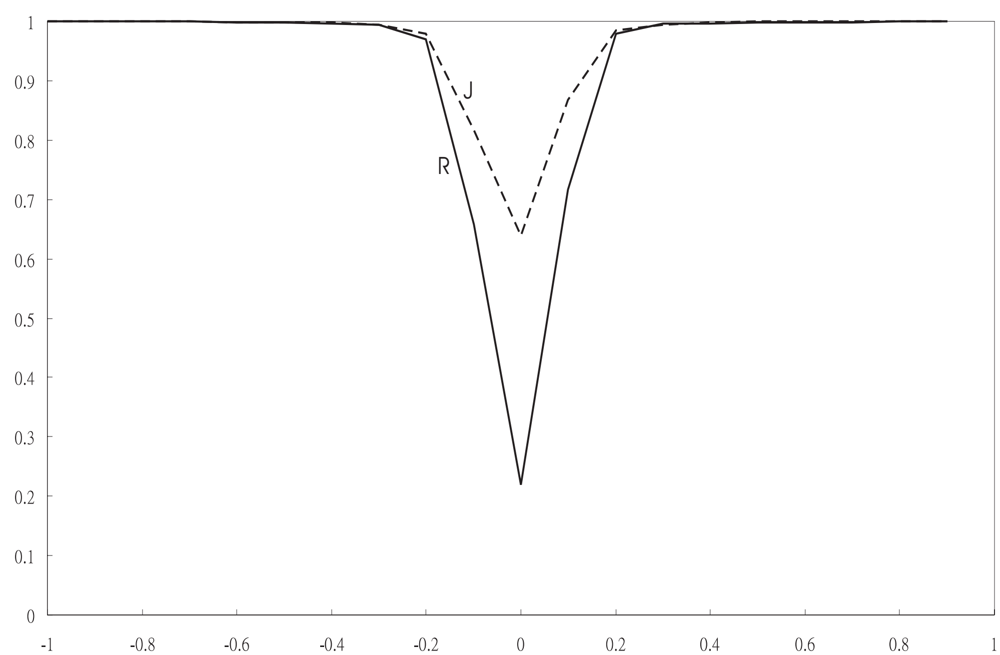

Figure 8: Finite sample powers under t_2 distribution and $p = 2$ and $q = 7$.

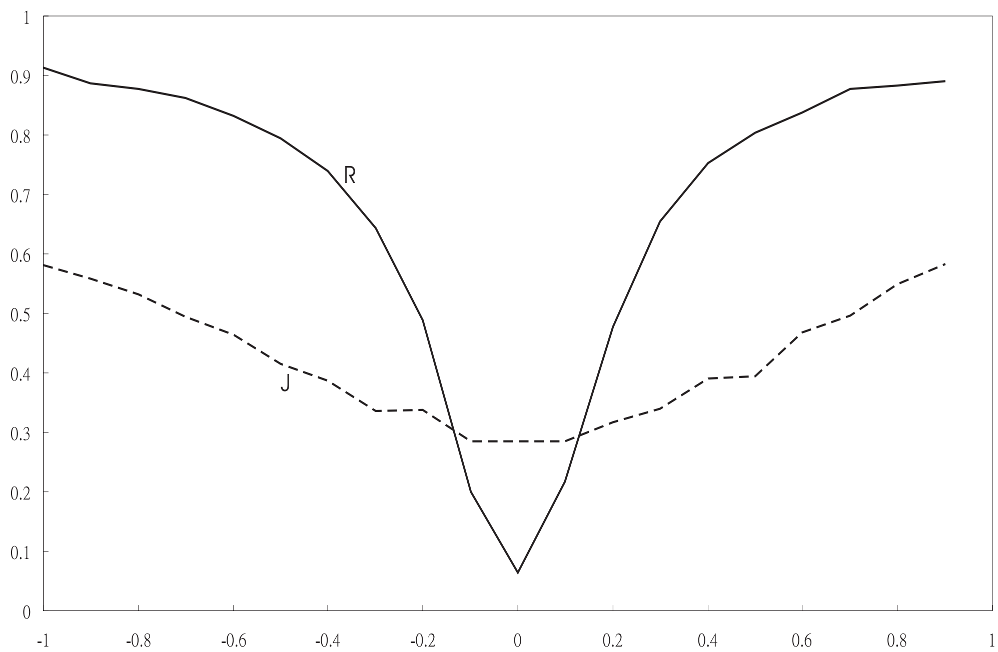

Figure 9: Finite sample powers under Cauchy distribution and $p = 3$ and $q = 3$.

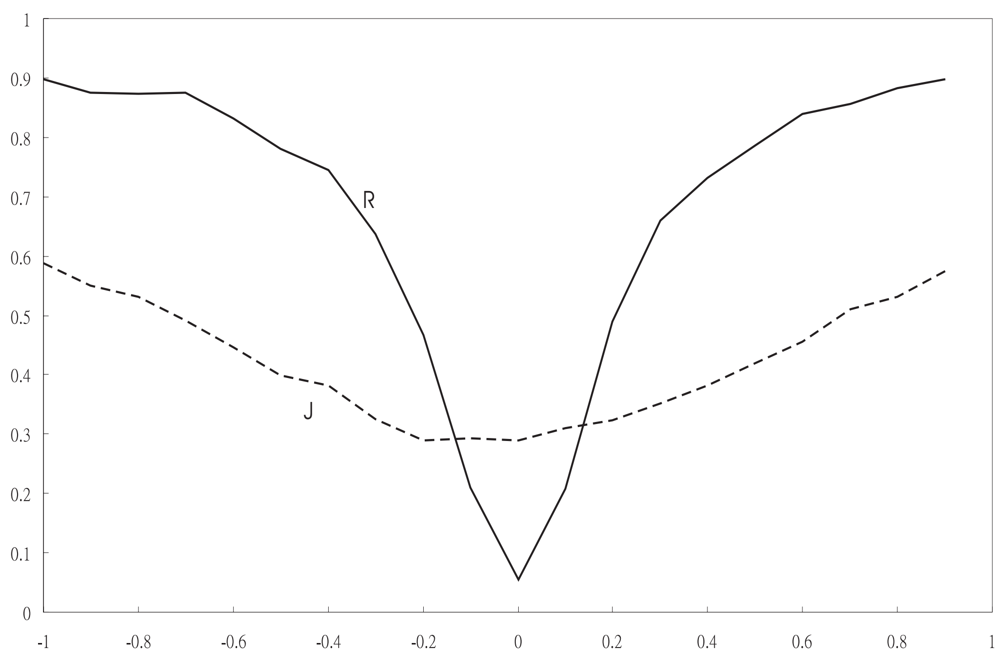

Figure 10: Finite sample powers under Cauchy distribution and $p = 6$ and $q = 3$.

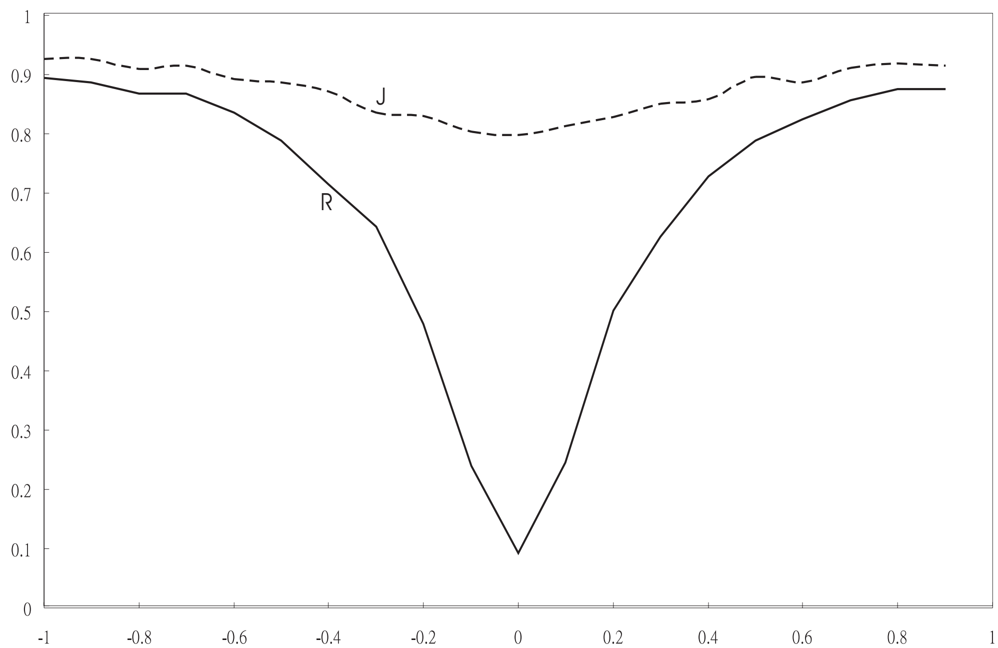

Figure 11: Finite sample powers under Cauchy distribution and $p = 2$ and $q = 7$.