

國立政治大學應用數學系

碩士學位論文

指導教授：李明融 博士

共同指導教授：謝宗翰 博士

**On the Nonlinear Differential Equation**

$$t^2 u'' = u^p$$

碩士班學生：姚信宇 撰

中華民國一百零一年七月

## 謝辭

本論文能夠順利完成，首先最要感謝指導教授李明融 副教授除了在專業知識上不斷給予我指導之外，在撰寫論文的細節上也給予我許多協助；另外也十分感謝逢甲大學航太系謝宗翰 副教授在這段時間持續寫信提供我數學上的物理意義、編排論文的注意事項以及時時叮嚀鼓勵我。

其實在完成這篇論文的過程中，我的逃避及放棄心態，讓原本應該按部就班就能夠完成的工作，在很匆促的時間內變得十分複雜和棘手。這真的必須再次感謝我的指導教授李明融 副教授，不僅協助我推導數學式及作圖，並且從老師身上讓我體會到身為一個「研究生」所應具備的精神—就是永不放棄及堅持到底！在寫論文的階段中，大家可能多少都會遇到挫折，而每個人克服困難的方式也不盡相同，而我很慶幸遇到如此好的兩位指導教授，他們所教導我的，並不是只有在寫論文上，更是教導我今後遇到任何難題所應該具有的精神與態度！

此外，就在我陷入最低潮之際，我很幸運遇到很多願意協助我的師長以及朋友，謝謝蔡炎龍 系主任給予我當頭棒喝，不給我休學的機會並提供我昂貴的數學軟體，讓我能最後的階段更加努力衝刺完成論文；也感謝陳天進 教授和陳政輝 助理教授，在這學期的修課中幫忙我許多；系辦的兩位助教，琬婷、偉慈，也謝謝妳們一直以來在生活上及規章中，不斷地給予我叮嚀與協助。另外，也感謝自強一二三舍的輔導員邱俊榮 老師，提供我一個暫時性的住宿環境，讓我在定稿時能無後顧之憂；還有謝謝經濟系博士班的起銓學長，應數系博士班的天財學長，給予我在寫這篇論文上的許多協助及建議；當然也感謝我的舍顧夥伴們，俊賢、君灝、昆諭及力航，不斷地為我加油打氣。

最重要的，我必須特別感謝我的家人，讓我有機會在政治大學就讀、學習以及成長；同時必須深深地向你們說聲抱歉，讓你們備受煎熬、等待許久。最後，還要感謝我最親愛的正妹·小~~~寶貝老婆，曉藍，謝謝妳不曾離棄我，給予我一個平靜的心情去寫完這篇碩士論文。

## 摘要

回顧一個重要的非線性二階方程式

$$\frac{d}{dt} \left( t^p \frac{du}{dt} \right) \pm t^\sigma u^n = 0,$$

這個方程式有許多有趣的物理應用，以 **Emden** 方程式的形式發生在天體物理學中；也以 **Fermi-Thomas** 方程式的形式出現在原子物理內。對於此類型的非線性方程式可以用來更頻繁且深入的探討數學物理，雖然目前仍存在著些許不確定性，不過如果在未來能有更全面的了解，這將有助於用來決定物理解的性質。

在這篇論文當中，我們討論微分方程式

$$t^2 u'' = u^p, \quad p \in \mathbb{N} - \{1\},$$

其正解的性質。這個方程式是著名的 **Emden-Fowler** 方程式的一種特殊情形，我們可以得到其解的一些有趣的現象。我們主要的結果是：

(a)  $u_1 = 0, u_0 > 0$ :

The life-span  $T^* \leq \exp(k_1)$ , where  $k_1 := s_0 + \frac{2(p+3)}{8-\varepsilon} \cdot \frac{2}{p-1} v(s_0)^{\frac{1-p}{2}}$ .

(b)  $u_1 > 0, u_0 > 0$ :

(i)  $E(0) \geq 0$ , the life-span  $T^* \leq \exp(k_2)$ , where  $k_2 := \frac{2}{p-1} \sqrt{\frac{p+1}{2}} u_0^{\frac{1-p}{2}}$ .

(ii)  $E(0) < 0$ , the life-span  $T^* \leq \exp(k_3)$ , where  $k_3 := \frac{2}{p-1} \frac{u_0}{u_1}$ .

(c)  $u_1 < 0, u_0 \in \left( 0, (-u_1)^{\frac{1}{p}} \right)$ :  $u(t) \leq \left( u_0 - (u_1 + u_0^p) \right) + (u_1 + u_0^p) t - u_0^p \ln t$ .

Furthermore, for  $E(0) \geq 0$ ,

$$u(t) \leq \left( u_0^{\frac{1-p}{2}} + \frac{p-1}{2} \sqrt{\frac{2}{p+1}} \ln t \right)^{\frac{2}{1-p}}.$$

**關鍵詞**：正解的爆炸時間、正解的最大存在時間、Emden-Fowler 方程式

## Abstract

Recall the important nonlinear second-order equation

$$\frac{d}{dt} \left( t^p \frac{du}{dt} \right) \pm t^\sigma u^n = 0,$$

this equation has several interesting physical applications, occurring in astrophysics in the form of the Emden equation and in atomic physics in the form of the Fermi-Thomas equation. These seems a little doubt that nonlinear equations of this type would enter with greater frequency into mathematical physics, were it more widely known with what ease the properties of the physical solutions can be determined.

In this paper we discuss the property of positive solution of the ordinary differential equation

$$t^2 u'' = u^p \quad \text{for } p \in \mathbb{N} - \{1\},$$

this equation is a special case of the well-known Emden-Fowler equation, we obtain some interesting phenomena for solutions. Our main results are:

(a)  $u_1 = 0, u_0 > 0$ :

The life-span  $T^* \leq \exp(k_1)$ , where  $k_1 := s_0 + \frac{2(p+3)}{8-\varepsilon} \cdot \frac{2}{p-1} v(s_0)^{\frac{1-p}{2}}$ .

(b)  $u_1 > 0, u_0 > 0$ :

(i)  $E(0) \geq 0$ , the life-span  $T^* \leq \exp(k_2)$ , where  $k_2 := \frac{2}{p-1} \sqrt{\frac{p+1}{2}} u_0^{\frac{1-p}{2}}$ .

(ii)  $E(0) < 0$ , the life-span  $T^* \leq \exp(k_3)$ , where  $k_3 := \frac{2}{p-1} \frac{u_0}{u_1}$ .

(c)  $u_1 < 0, u_0 \in \left( 0, (-u_1)^{\frac{1}{p}} \right)$ :  $u(t) \leq \left( u_0 - (u_1 + u_0^p) \right) + (u_1 + u_0^p)t - u_0^p \ln t$ .

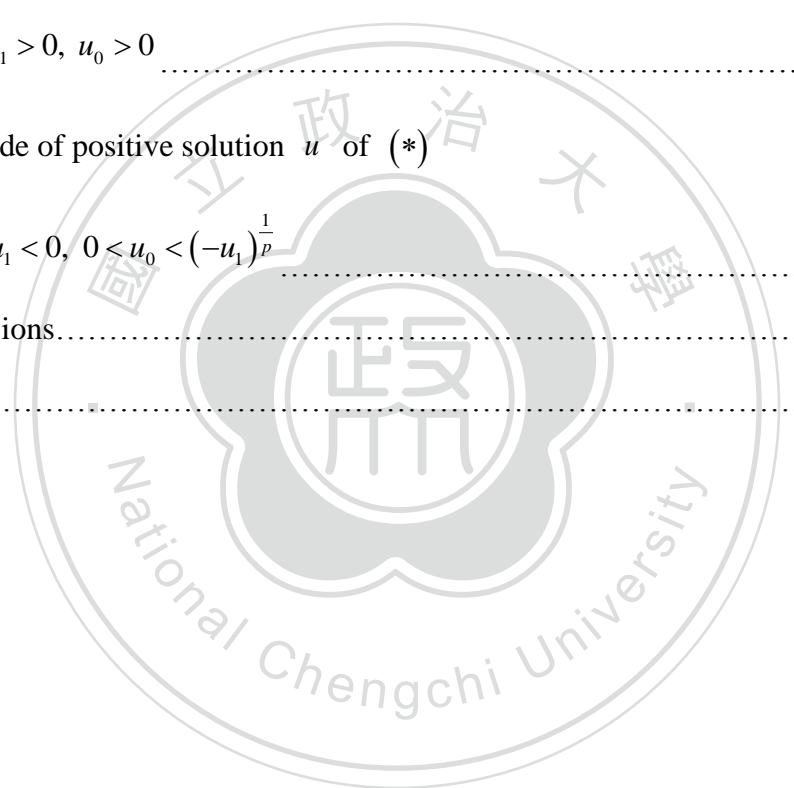
Furthermore, for  $E(0) \geq 0$ ,

$$u(t) \leq \left( u_0^{\frac{1-p}{2}} + \frac{p-1}{2} \sqrt{\frac{2}{p+1}} \ln t \right)^{\frac{2}{1-p}}.$$

**Keywords** : blow-up time for positive solution 、 the life-span for positive solution 、 Emden-Fowler equation

# Contents

1. Introduction.....	1
2. Local Existence of Solutions.....	3
3. Notation and Fundamental Lemmas.....	6
4. Estimates for the life-span of positive solution $u$ of (*) under $u_1 = 0, u_0 > 0$ .....	11
5. Estimates for the life-span of positive solution $u$ of (*) under $u_1 > 0, u_0 > 0$ .....	22
6. Magnitude of positive solution $u$ of (*) under $u_1 < 0, 0 < u_0 < (-u_1)^{\frac{1}{p}}$ .....	32
7. Conclusions.....	43
References.....	44



## Contents of Graphs

Figure 1 Graph of $u$ .....	15
Figure 2 Graph of $u$ .....	16
Figure 3 Graph of $u$ .....	16
Figure 4 Graph of $u$ .....	17
Figure 5 Graph of $u$ .....	17
Figure 6 Graph of $u$ .....	18
Figure 7 Graph of $u$ .....	18
Figure 8 Graph of $u$ .....	19
Figure 9 Graph of $u$ .....	19
Figure 10 Graph of $u$ .....	20
Figure 11 Graph of $u$ .....	20
Figure 12 Graph of $u$ .....	21
Figure 13 Graph of $u$ .....	24
Figure 14 Graph of $u$ .....	25
Figure 15 Graph of $u$ .....	25
Figure 16 Graph of $u$ .....	26
Figure 17 Graph of $u$ .....	26
Figure 18 Graph of $u$ .....	27
Figure 19 Graph of $u$ .....	27
Figure 20 Graph of $u$ .....	28
Figure 21 Graph of $u$ .....	28
Figure 22 Graph of $u$ .....	29
Figure 23 Graph of $u$ .....	29
Figure 24 Graph of $u$ .....	30
Figure 25 Graph of $u$ .....	30

Figure 26 Graph of $u$ .....	31
Figure 27 Graph of $u$ .....	31
Figure 28 Graph of $f(t, u_0)$ .....	33
Figure 29 Graph of $f(t, u_0)$ .....	34
Figure 30 Graph of $f(t, u_0)$ .....	34
Figure 31 Graph of $f(t, u_0)$ .....	35
Figure 32 Graph of $f(t, u_0)$ .....	35
Figure 33 Graph of $f(t, u_0)$ .....	36
Figure 34 Graph of $f(t, u_0)$ .....	36
Figure 35 Graph of $f(t, u_0)$ .....	37
Figure 36 Graph of $f(t, u_0)$ .....	37
Figure 37 Graph of $f(t, p)$ .....	39
Figure 38 Graph of $f(t, p)$ .....	40
Figure 39 Graph of $f(t, p)$ .....	40
Figure 40 Graph of $g(u_0, t)$ .....	41
Figure 41 Graph of $g(u_0, t)$ .....	41
Figure 42 Graph of $g(u_0, t)$ .....	42
Figure 43 Graph of $g(u_0, t)$ .....	42